Linear Algebra: Homework 6

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Section 3.2

Exercise 1

Solve the equation for X, given that:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} X - 2A + 3B &= 0 \\ X &= 2A - 3B \\ &= \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 3 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 \\ 3 & 5 \end{bmatrix} \end{aligned}$$

Exercise 3

Solve the equation for X, given that:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} 2(A + 2B) &= 3X \\ X &= \frac{2(A+2B)}{3} \\ &= \frac{2}{3}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 2 & 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} \\ \frac{10}{3} & \frac{12}{3} \end{bmatrix} \end{aligned}$$

Exercise 5

Write B as a linear combination of the other matrices if possible.

$$B = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$B = 2A_1 + A_2$$

Exercise 7

Write B as a linear combination of the other matrices if possible.

$$B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Not possible.

Exercise 9

Find the general form of the span of the indicated matrices.

$$A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\begin{aligned}
span(A_1, A_2) &= c_1 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_1 & 2c_1 + c_2 \\ -c_1 + 2c_2 & c_1 + c_2 \end{bmatrix} \\
&= \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\
\left[\begin{array}{cc|c} 1 & 0 & w \\ 2 & 1 & x \\ -1 & 2 & y \\ 1 & 1 & z \end{array}\right] &= \begin{bmatrix} 1 & 0 & w \\ 1 & 0 & x - z \\ 0 & 2 & y + w \\ 1 & 1 & z \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & w \\ 1 & 0 & x - z \\ 0 & 1 & \frac{y+w}{2} \\ 0 & 1 & z - w \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & w \\ 0 & 0 & x - z - w \\ 0 & 1 & \frac{y+w}{2} \\ 0 & 0 & z - w - \frac{y+w}{2} \end{bmatrix} \\
w &= x - z \\
z - w &= \frac{y + w}{2} \\
2(z - w) &= y + w \\
2z - 3w &= y \\
2z - 3(x - z) &= y \\
5z - 3x &= y \\
span(A_1, A_2) &= \begin{bmatrix} x - z & x \\ 5z - 3x & z \end{bmatrix}
\end{aligned}$$

Exercise 11

Find the general form of the span of the indicated matrices.

$$A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
span(A_1, A_2, A_3) &= c_1A_1 + c_2A_2 + c_3A_3 \\
&= \begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} 1 & -1 & 1 & u \\ 0 & 2 & 1 & v \\ -1 & 0 & 1 & w \\ 0 & 0 & 0 & x \\ 1 & 1 & 0 & y \\ 0 & 0 & 0 & z \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 1 & u \\ 1 & 1 & 0 & y \\ 0 & 2 & 1 & v \\ -1 & 0 & 1 & w \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & z \end{bmatrix} \\
&= \begin{bmatrix} 0 & -2 & 1 & u - y \\ 0 & 2 & 1 & v \\ 1 & 1 & 0 & y \\ -1 & 0 & 1 & w \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & z \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 2 & u - y + v \\ 0 & 1 & 0 & v - y - w \\ 0 & 2 & 2 & 2y + 2w \\ -1 & 0 & 1 & w \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & z \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 2 & u - y + v \\ 0 & 1 & 0 & v - y - w \\ 0 & 0 & 0 & 2y + 2w - u + y - v - 2(v - y - w) \\ -1 & 0 & 1 & w \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & z \end{bmatrix} \\
0 &= 2y + 2w - u + y - v - 2(v - y - w) \\
0 &= 5y + 4w - u - 3v \\
span(A_1, A_2, A_3) &= \begin{bmatrix} 5y + 4w - 3v & \frac{5y+4w-u}{3} & \frac{u+3v-5y}{4} \\ x & \frac{u+3v-4w}{5} & z \end{bmatrix}
\end{aligned}
$$

Exercise 13

Determine whether the given matrices are linearly independent.

$$
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}
$$

$$\begin{bmatrix} c_1 + 4c_2 & 2c_1 + 3c_2 \\ 3c_1 + 2c_2 & 4c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 0 \\ 4 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ -10 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The only solution for this is $c_1 = c_2 = 0$, so the matrices are linearly independent.

Exercise 15

Determine whether the given matrices are linearly independent.

$$\begin{bmatrix} 0 & 1 \\ 5 & 2 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & -1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & -3 \\ 1 & 9 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0c_1 + 1c_2 - 2c_3 - 1c_4 & 1c_1 + 0c_2 - 1c_3 - 3c_4 \\ 5c_1 + 2c_2 + 0c_3 + 1c_4 & 2c_1 + 3c_2 + 1c_3 + 9c_4 \\ -1c_1 + 1c_2 + 0c_3 + 4c_4 & 0c_1 + 1c_2 + 2c_3 + 5c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 & -1 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 9 & 0 \\ -1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The only solution for this is $c_1 = c_2 = c_3 = c_4 = 0$, so the matrices are linearly independent.

Exercise 22

Prove that, for square matrices A and B, $AB = BA$ if and only if $(A - B)(A + B) = A^2 - B^2$.

$$\begin{aligned} \text{Assume } AB &= BA \\ AB - BA &= 0 \\ (A - B)(A + B) &= A^2 + AB - BA - B^2 \\ (A - B)(A + B) &= A^2 + 0 - B^2 \\ &= A^2 - B^2 \end{aligned}$$

Converse:

$$\begin{aligned} \text{Assume } (A - B)(A + B) &= A^2 - B^2 \\ A^2 - AB - BA - B^2 &= A^2 - B^2 \\ AB - BA &= 0 \\ AB &= BA \end{aligned}$$

Exercise 29

Prove that the product of two upper triangular $n \times n$ matrices is upper triangular.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{nn} \end{bmatrix}$$

$$AB = \begin{bmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \sum_{i=1}^n a_{1i}b_{i2} & \dots & \sum_{i=1}^n a_{1i}b_{in} \\ \sum_{i=1}^n a_{2i}b_{i1} & \sum_{i=1}^n a_{2i}b_{i2} & \dots & \sum_{i=1}^n a_{2i}b_{in} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^n a_{ni}b_{i1} & \sum_{i=1}^n a_{ni}b_{i2} & \dots & \sum_{i=1}^n a_{ni}b_{in} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & \sum_{i=1}^n a_{1i}b_{i2} & \dots & \sum_{i=1}^n a_{1i}b_{in} \\ 0 & a_{22}b_{22} & \dots & \sum_{i=1}^n a_{2i}b_{in} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{nn}b_{nn} \end{bmatrix}$$

Exercise 30

Prove Theorem 3.4(a)-(c).

(a) $(A^T)^T = A$

$$\begin{aligned} B &= A^T \\ \therefore b_{ij} &= a_{ji} \\ C &= B^T = (A^T)^T \\ \therefore c_{ij} &= b_{ji} \\ \therefore a_{ij} &= b_{ji} = c_{ij} \\ A &= C = (A^T)^T \end{aligned}$$

(b) $(A+B)^T = A^T + B^T$

$$\begin{aligned} (A+B)^T &= C \\ a_{ij} + b_{ij} &= c_{ij} \\ A^T + B^T &= C \\ \therefore A^T + B^T &= (A+B)^T \end{aligned}$$

(c) $(kA)^T = k(A^T)$

$$\begin{aligned}
B &= A^T \\
\therefore b_{ij} &= a_{ji} \\
kB &= k(A^T) \\
kb_{ij} &= k(a_{ji}) \\
&= (kA)^T \\
k(B^T) &= k(A^T) = (kA)^T
\end{aligned}$$

Exercise 31

Prove Theorem 3.4(e).

$$(A^r)^T = (A^T)^r \text{ for all nonnegative integers } r$$

Base Case ($r = 1$):

$$(A^1)^T = A^T = (A^T)^1$$

Induction Hypothesis:

$$(A^r)^T = (A^T)^r \text{ for } r \geq 1$$

Induction:

$$\begin{aligned}
(A^{r+1})^T &= (A^r A^1)^T \\
&= (A^r)^T (A^1)^T \\
&= (A^T)^r (A^T)^1 \text{ by the base case and induction hypothesis} \\
&= (A^T)^{r+1}
\end{aligned}$$

Exercise 32

Using induction, prove that for all $n \geq 1$, $(A_1 + A_2 + \cdots + A_n)^T = A_1^T + A_2^T + \cdots + A_n^T$.
Base Case ($n = 1$):

$$(A_1)^T = A_1^T$$

Induction Hypothesis:

$$(A_1 + A_2 + \cdots + A_n)^T = A_1^T + A_2^T + \cdots + A_n^T$$

Induction:

$$\begin{aligned}
(A_1 + A_2 + \cdots + A_n + A_{n+1})^T &= (A_1 + A_2 + \cdots + A_n)^T + A_{n+1}^T \text{ by Theorem 3.4b} \\
&= A_1^T + A_2^T + \cdots + A_n^T + A_{n+1}^T \text{ by the induction hypothesis}
\end{aligned}$$

Exercise 33

Using induction, prove that for all $n \geq 1$, $(A_1 A_2 \ldots A_n)^T = A_n^T \ldots A_2^T A_1^T$.
Base Case ($n = 1$):

$$(A_1)^T = A_1^T$$

Induction Hypothesis:

$$(A_1 A_2 \ldots A_n)^T = A_n^T \ldots A_2^T A_1^T$$

Induction:

$$\begin{aligned}
(A_1 A_2 \ldots A_n A_{n+1})^T &= ((A_1 A_2 \ldots A_n) A_{n+1})^T \\
&= A_{n+1}^T (A_1 A_2 \ldots A_n)^T \text{ by Theorem 3.4d} \\
&= A_{n+1}^T A_n^T \ldots A_2^T A_1^T \text{ by the induction hypothesis}
\end{aligned}$$

Exercise 34

Prove Theorem 3.5(b): For any matrix A, AA^T and A^TA are symmetric matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

$$AA^T = A^TA = \begin{bmatrix} \sum_{i=1}^n a_{1i}a_{1i} & \sum_{i=1}^n a_{1i}a_{2i} & \dots & \sum_{i=1}^n a_{1i}a_{ni} \\ \sum_{i=1}^n a_{2i}a_{1i} & \sum_{i=1}^n a_{2i}a_{2i} & \dots & \sum_{i=1}^n a_{2i}a_{ni} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^n a_{ni}a_{1i} & \sum_{i=1}^n a_{ni}a_{2i} & \dots & \sum_{i=1}^n a_{ni}a_{ni} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^n a_{1i}a_{1i} & \sum_{i=1}^n a_{1i}a_{2i} & \dots & \sum_{i=1}^n a_{1i}a_{ni} \\ \sum_{i=1}^n a_{1i}a_{2i} & \sum_{i=1}^n a_{2i}a_{2i} & \dots & \sum_{i=1}^n a_{2i}a_{ni} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^n a_{1i}a_{ni} & \sum_{i=1}^n a_{2i}a_{ni} & \dots & \sum_{i=1}^n a_{ni}a_{ni} \end{bmatrix}$$

$$\sum_{i=1}^n a_{1i}a_{i1} = \sum_{i=1}^n a_{i1}a_{1i} \text{ and so on...}$$

Exercise 37

Which of the following matrices are skew-symmetric?

(a) $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ Diagonal does not contain zeroes. Not skew-symmetric.

(b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A^T = -A$. This matrix is skew-symmetric.

(c) $A = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$ $A^T = -A$. This matrix is skew-symmetric.

(d) $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix}$

$$\begin{aligned} A^T &= \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix} \\ -A &= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -5 \\ -2 & -5 & 0 \end{bmatrix} \\ A^T &\neq -A \end{aligned}$$

This matrix is not skew-symmetric.

Exercise 39

Prove that the main diagonal of a skew-symmetric matrix must consist entirely of zeros.
Let A be a skew-symmetric matrix.

$$\begin{aligned} a_{ij} &= -aji \\ i &= j \text{ along the diagonal} \\ a_{ii} &= -a_{ii} \\ 2a_{ii} &= 0 \\ a_{ii} &= 0 \end{aligned}$$

Exercise 40

Prove that if A and B are skew-symmetric $n \times n$ matrices, then so is $A + B$.

$$\begin{aligned} a_{ij} &= -a_{ji} \\ b_{ij} &= -b_{ji} \\ \text{Let } C &= A + B \\ c_{ij} &= a_{ij} + b_{ij} \\ &= -a_{ji} - b_{ji} \\ &= -(a_{ji} + b_{ji}) \\ &= -c_{ji} \end{aligned}$$

Exercise 41

If A and B skew-symmetric 2×2 matrices, under what conditions is AB skew-symmetric?

$$\begin{aligned} A &= \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \\ AB &= \begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix} \\ -ab &= 0 \\ \therefore a &= 0 \text{ or } b = 0 \end{aligned}$$

Either A or B must be filled with zeros.

Exercise 42

Prove that if A is an $n \times n$ matrix, then $A - A^T$ is skew-symmetric.

$$\begin{aligned}(A - A^T)^T &= A^T - (A^T)^T \\ &= A^T - A \\ &= (-1)(A - A^T)\end{aligned}$$

Therefore $A - A^T$ is skew-symmetric.

Exercise 43

(a) Prove that any square matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

$$\begin{aligned}A &= A + A^T - A^T \\ &= A + \frac{1}{2}A^T - \frac{1}{2}A^T \\ &= \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T \\ &= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)\end{aligned}$$

$A + A^T$ and $A - A^T$ are symmetric and skew-symmetric matrices, respectively.

(b) Illustrate part (a) for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right)$$

Exercise 44

If A and B are $n \times n$ matrices, prove the following properties of the trace:

(a) $tr(A + B) = tr(A) + tr(B)$

$$\begin{aligned}tr(A + B) &= \sum_{i=1}^{n}(a_{ii} + b_{ii}) \\ &= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\ &= tr(A) + tr(B)\end{aligned}$$

(b) $tr(kA) = k\ tr(A)$

$$\begin{aligned}tr(kA) &= \sum_{i=1}^{n}(ka_{ii}) \\ &= k\left(\sum_{i=1}^{n} a_{ii}\right) \\ &= k\ tr(A)\end{aligned}$$

Exercise 45

Prove that if A and B are $n \times n$ matrices, then $tr(AB) = tr(BA)$.

$$\begin{aligned}
tr(AB) &= \sum_{i=1}^{n} row_i(A) \cdot col_i(B) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} b_{ji} \right) \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_{ji} a_{ij} \right) \\
&= tr(BA)
\end{aligned}$$

Exercise 46

If A is any matrix, to what is $tr(AA^T)$ equal?

$$\begin{aligned}
C &= AA^T \\
c_{ij} &= \sum_{i=1}^{n} row_i(A) \cdot col_i(A^T) \\
&= \sum_{i=1}^{n} row_i(A) \cdot row_i(A) \\
tr(AA^T) &= \sum_{i=1}^{n} \|row_i(A)\|^2
\end{aligned}$$

Exercise 47

Show that there are no 2×2 matrices A and B such that $AB - BA = I_2$.

$$\begin{aligned}
tr(AB - BA) &= tr(AB + (-1)BA) \\
&= tr(AB) + tr((-1)BA) \\
&= tr(AB) - tr(BA) \\
&= 0 \\
\therefore AB - BA &= \begin{bmatrix} 0 & a_{11}b_{12} + a_{12}b_{22} - b_{11}a_{12} + b_{12}a_{22} \\ a_{21}b_{11} + a_{22}b_{21} - b_{21}a_{11} + b_{22}a_{21} & 0 \end{bmatrix} \\
&\neq I_2
\end{aligned}$$

Section 3.3

Exercise 1

Find the inverse of the given matrix (if it exists) using Theorem 3.8.

$$A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$$

$$ad - bc = 1$$

$$A' = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$$

Exercise 3

Find the inverse of the given matrix (if it exists) using Theorem 3.8.

$$A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

$$ad - bc = 0$$

The matrix is not invertible.

Exercise 5

Find the inverse of the given matrix (if it exists) using Theorem 3.8.

$$A = \begin{bmatrix} \frac{3}{4} & \frac{3}{5} \\ \frac{5}{6} & \frac{2}{3} \end{bmatrix}$$

$$\begin{aligned} ad - bc &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

The matrix is not invertible.

Exercise 7

Find the inverse of the given matrix (if it exists) using Theorem 3.8.

$$A = \begin{bmatrix} -1.5 & -4.2 \\ 0.5 & 2.4 \end{bmatrix}$$

$$\begin{aligned} ad - bc &= -3.6 - (-2.1) \\ &= -1.5 \\ A' &= \begin{bmatrix} \frac{2.4}{-1.5} & -\frac{-4.2}{-1.5} \\ -\frac{0.5}{-1.5} & \frac{-1.5}{-1.5} \end{bmatrix} \\ &= \begin{bmatrix} -1.6 & -2.8 \\ 0.33 & 1 \end{bmatrix} \end{aligned}$$

Exercise 9

Find the inverse of the given matrix (if it exists) using Theorem 3.8.

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\begin{aligned} \frac{1}{\det(A)} &= \frac{1}{a^2 + b^2} \\ A' &= \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix} \end{aligned}$$

Exercise 11

Solve the given system using the method of Example 3.25.

$$\begin{aligned}
2x + y &= -1 \\
5x + 3y &= 2 \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \left(\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
&= \frac{1}{6-5} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -5 \\ 9 \end{bmatrix}
\end{aligned}$$

Exercise 18

By induction, prove that if $A_1, A_2, \ldots, A_n$ are invertible matrices of the same size, then the product $A_1 A_2 \ldots A_n$ is invertible and $(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1} \ldots A_2^{-1} A_1^{-1}$.
Base Case:

$$(A_1 A_2)^{-1} = A_2^{-1} A_1^{-1}$$

Induction Hypothesis:

$$(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1} \ldots A_2^{-1} A_1^{-1}$$

Induction:

$$\begin{aligned}
(A_1 A_2 \ldots A_n A_{n+1})^{-1} &= ((A_1 A_2 \ldots A_n)(A_{n+1}))^{-1} \\
&= A_{n+1}^{-1} (A_1 A_2 \ldots A_n)^{-1} \text{ by Theorem 3.9c} \\
&= A_{n+1}^{-1} A_n^{-1} \ldots A_2^{-1} A_1^{-1} \text{ by the induction hypothesis}
\end{aligned}$$

Exercise 44

A square matrix A is called **idempotent** if $A^2 = A$.

(a) Find three idempotent 2×2 matrices.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) Prove that the only invertible idempotent $n \times n$ matrix is the identity matrix.

$$\begin{aligned}
A &= A^2 \\
A(A^{-1}) &= A^2(A^{-1}) \\
I_n &= AAA^{-1} \\
I_n &= AI_n \\
A &= I_n
\end{aligned}$$

Exercise 45

Show that if A is a a square matrix that satisfies the equation $A^2 - 2A + I = 0$, then $A^{-1} = 2I - A$.

$$\begin{aligned}
A^2 - 2A + I &= 0 \\
A^2(A^{-1}) - 2A(A^{-1}) + I(A^{-1}) &= 0(A^{-1}) \\
AI - 2(I) + A^{-1} &= 0 \\
A - 2I + A^{-1} &= 0 \\
A^{-1} &= 2I - A
\end{aligned}$$

Exercise 46

Prove that if a symmetric matris is invertible, then its inverse is symmetric also.

$$\begin{aligned}
A &= A^T \\
(A^{-1})^T &= (A^T)^{-1} \\
(A^{-1})^T &= A^{-1}
\end{aligned}$$

Since the transpose of the inverse is equal to the inverse, the inverse must be symmetric.

Exercise 47

Prove that if A and B are square matrices and AB is invertible, then both A and B are invertible.

$$\begin{aligned}
B\vec{x} &= 0 \\
(A)B\vec{x} &= (A)0 \\
AB\vec{x} &= 0 \\
\vec{x} &= 0 \quad \text{Theorem 3.12c} \\
&\therefore B \text{ is invertible} \\
A &= AI \\
&= A(BB^{-1}) \\
&= (AB)(B^{-1})
\end{aligned}$$

Since A is the product of two invertible matrices, A is invertible as well.

Exercise 49

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

$$\begin{aligned}
\begin{bmatrix} -2 & 4 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 10 & 0 & 1 & 4 \\ 3 & -1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \frac{1}{10} & \frac{2}{5} \\ -3 & 1 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 1 & \frac{3}{10} & \frac{1}{5} \end{bmatrix}
\end{aligned}$$

Exercise 51

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

$$\begin{aligned}
\begin{bmatrix} 1 & a & 1 & 0 \\ -a & 1 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & a & 1 & 0 \\ a^2 & -a & 0 & -a \end{bmatrix} \\
&= \begin{bmatrix} 1+a^2 & 0 & 1 & -a \\ a^2 & -a & 0 & -a \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \frac{1}{1+a^2} & \frac{-a}{1+a^2} \\ 1 & \frac{-1}{a} & 0 & \frac{-1}{a} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \frac{1}{1+a^2} & \frac{-a}{1+a^2} \\ 0 & \frac{-1}{a} & -\frac{1}{1+a^2} & \frac{-1}{a} + \frac{a}{1+a^2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \frac{1}{1+a^2} & \frac{-a}{1+a^2} \\ 0 & 1 & \frac{a}{1+a^2} & 1 + \frac{-a^2}{1+a^2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \frac{1}{1+a^2} & \frac{-a}{1+a^2} \\ 0 & 1 & \frac{a}{1+a^2} & \frac{1}{1+a^2} \end{bmatrix}
\end{aligned}$$

Exercise 53

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -4 & 1 & 0 \\ 0 & 5 & -5 & -2 & 0 & 1 \end{bmatrix}$$

No inverse exists since the last two rows can cancel each other out.

Exercise 55

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

$$\begin{aligned}
\begin{bmatrix} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & a & 0 & -\frac{1}{a} & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a^2} & \frac{1}{a} & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a^2} & \frac{1}{a} & 0 \\ 0 & 0 & a & \frac{1}{a^2} & -\frac{1}{a} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a^2} & \frac{1}{a} & 0 \\ 0 & 0 & 1 & \frac{1}{a^3} & -\frac{1}{a^2} & \frac{1}{a} \end{bmatrix}
\end{aligned}$$

Exercise 57

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

$$
\begin{aligned}
\begin{bmatrix} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}
&= \begin{bmatrix} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -4 & 2 & 2 & 1 & -2 & 0 \\ 0 & 0 & -3 & 2 & 3 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -4 & 2 & 2 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 & -9 & -2 & 4 & -3 \\ 0 & 0 & 1 & -1 & -4 & -1 & 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -4 & 2 & 2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -4 & 2 & 2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 1 & -4 & 2 & 2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 1 & -4 & 2 & 2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & -11 & -2 & 5 & -4 \\ 0 & 1 & 0 & 0 & 4 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{bmatrix}
\end{aligned}
$$

Exercise 59

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ a & b & c & d & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & b & c & d & -a & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d & -a & -b & -c & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{a}{d} & -\frac{b}{d} & -\frac{c}{d} & \frac{1}{d} \end{array}\right]$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech