

Linear Algebra: Homework 3

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Section 1.2

Exercise 48

Find all values of the scalar k for which the two vectors are orthogonal.

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} k+1 \\ k-1 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = 0$$

$$2(k+1) + 3(k-1) = 0$$

$$2k + 2 + 3k - 3 = 0$$

$$5k = 1$$

$$k = \frac{1}{5}$$

Exercise 49

Find all values of the scalar k for which the two vectors are orthogonal.

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = 0$$

$$k^2 - k - 6 = 0$$

$$(k-3)(k+2) = 0$$

$$k = 3 \quad k = -2$$

Exercise 50

Describe all vectors that are orthogonal to $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$3x + y = 0$$

They are all parallel to the line described by $y = -3x$.

Exercise 51

Describe all vectors that are orthogonal to $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$.

They are all parallel to the line described by $ax + by = 0$.

Exercise 52

Under what conditions are the following true for vectors \vec{u} and \vec{v} in \mathbb{R}^2 or \mathbb{R}^3 ?

1. $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$: This is true when the vectors are parallel.
2. $\|\vec{u} + \vec{v}\| = \|\vec{u}\| - \|\vec{v}\|$: This is true when the vectors are antiparallel.

Exercise 53

Prove Theorem 1.2(b).

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

Proof:

$$\begin{aligned} \vec{u} \cdot (\vec{v} + \vec{w}) &= \sum_{i=1}^n u_i(v_i + w_i) \\ &= \sum_{i=1}^n (u_i v_i + u_i w_i) \\ &= \sum_{i=1}^n u_i v_i + \sum_{i=1}^n u_i w_i \\ &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \end{aligned}$$

Exercise 54

Prove Theorem 1.2(d).

$$\begin{aligned} \vec{u} \cdot \vec{u} &\geq 0 \\ \vec{u} \cdot \vec{u} = 0 &\text{ iff } \vec{u} = \vec{0} \end{aligned}$$

Proof:

$$\begin{aligned} \vec{u} \cdot \vec{u} &= \sum_{i=1}^n u_i u_i \\ &= \sum_{i=1}^n (u_i)^2 \end{aligned}$$

$(u_i)^2$ is non-negative, therefore the summation must be greater than or equal to 0, and only equal to 0 when $\vec{u} = \vec{0}$.

Exercise 55

Prove the stated property of distance between vectors.

$$d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$$

Proof:

$$\begin{aligned}d(\vec{u}, \vec{v}) &= \sqrt{\sum_{i=1}^n (u_i - v_i)^2} \\&= \sqrt{\sum_{i=1}^n (-1)^2 (v_i - u_i)^2} \\&= \sqrt{\sum_{i=1}^n (v_i - u_i)^2} \\&= d(\vec{v}, \vec{u})\end{aligned}$$

Exercise 56

Prove the stated property of distance between vectors.

$$d(\vec{u}, \vec{w}) \leq d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w})$$

Proof:

$$\begin{aligned}d(\vec{u}, \vec{w}) &\leq d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w}) \\ \|\vec{u} - \vec{w}\| &\leq \|\vec{u} - \vec{v}\| + \|\vec{v} - \vec{w}\| \\ \|\vec{u} - \vec{w}\| &\leq \|\vec{u} - \vec{v} + \vec{v} - \vec{w}\| \quad (\text{By the triangle inequality}) \\ \|\vec{u} - \vec{w}\| &\leq \|\vec{u} - \vec{w}\|\end{aligned}$$

Exercise 57

Prove the stated property of distance between vectors.

$$d(\vec{u}, \vec{v}) = 0 \quad \text{iff} \quad \vec{u} = \vec{v}$$

Proof:

$$\begin{aligned}0 &= d(\vec{u}, \vec{v}) \\ 0 &= \sqrt{\sum_{i=1}^n (u_i - v_i)^2} \\ 0 &= \sum_{i=1}^n (u_i - v_i)^2 \\ 0 &= \sum_{i=1}^n (u_i - v_i) \\ 0 &= \sum_{i=1}^n u_i - \sum_{i=1}^n v_i \\ 0 &= \vec{u} - \vec{v} \\ \vec{v} &= \vec{u}\end{aligned}$$

Exercise 58

Prove that $\vec{u} \cdot c\vec{v} = c(\vec{u} \cdot \vec{v})$.

$$\begin{aligned}\vec{u} \cdot c\vec{v} &= \sum_{i=1}^n u_i(cv_i) \\ &= \sum_{i=1}^n c(u_iv_i) \\ &= c \sum_{i=1}^n u_iv_i \\ &= c(\vec{u} \cdot \vec{v})\end{aligned}$$

Exercise 59

Prove that $\|\vec{u} - \vec{v}\| \geq \|\vec{u}\| - \|\vec{v}\|$.

$$\|\vec{w} + \vec{v}\| \leq \|\vec{w}\| + \|\vec{v}\| \quad (\text{Triangle Inequality})$$

$$\text{Let : } \vec{w} = \vec{u} - \vec{v}$$

$$\|\vec{u} - \vec{v} + \vec{v}\| \leq \|\vec{u} - \vec{v}\| + \|\vec{v}\|$$

$$\|\vec{u}\| \leq \|\vec{u} - \vec{v}\| + \|\vec{v}\|$$

$$\|\vec{u}\| - \|\vec{v}\| \leq \|\vec{u} - \vec{v}\|$$

$$\|\vec{u} - \vec{v}\| \geq \|\vec{u}\| - \|\vec{v}\|$$

Exercise 60

Suppose know that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$. Does it follow that $\vec{v} = \vec{w}$? If it does, give a proof that is valid in \mathbb{R}^n . Otherwise, give a counterexample.

Suppose $\vec{u} = \vec{0}$. \vec{v} and \vec{w} can be any vector in \mathbb{R}^n .

$$\vec{0} \cdot \langle 1, 2 \rangle = \vec{0} \cdot \langle 3, 4 \rangle$$

Exercise 61

Prove that $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2$.

$$\begin{aligned}(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} \\ &= \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v} \\ &= \|\vec{u}\|^2 - \|\vec{v}\|^2\end{aligned}$$

Exercise 62a

Prove that $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$.

$$\begin{aligned}\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 &= (\|\vec{u}\|^2 + \|\vec{v}\|^2) + (\|\vec{u}\|^2 + \|\vec{-v}\|^2) \quad (\text{Pythagorean Theorem}) \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 + \|\vec{u}\|^2 + \|\vec{v}\|^2 \\ &= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2\end{aligned}$$

Exercise 63

Prove that $\vec{u} \cdot \vec{v} = \frac{1}{4}\|\vec{u} + \vec{v}\|^2 - \frac{1}{4}\|\vec{u} - \vec{v}\|^2$.

$$\begin{aligned} \frac{1}{4}\|\vec{u} + \vec{v}\|^2 - \frac{1}{4}\|\vec{u} - \vec{v}\|^2 &= \frac{1}{4}\sqrt{(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})}^2 - \frac{1}{4}\sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}^2 \\ &= \frac{1}{4}(\vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} - (\vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v})) \\ &= \vec{u} \cdot \vec{v} \end{aligned}$$

Exercise 64a

Prove that $\|\vec{u} + \vec{v}\| = \|\vec{u} - \vec{v}\|$ if and only if \vec{u} and \vec{v} are orthogonal.

$$\begin{aligned} \|\vec{u} + \vec{v}\| &= \|\vec{u} - \vec{v}\| \\ \sqrt{(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})} &= \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})} \\ (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} &= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ 2\vec{u} \cdot \vec{v} &= 0 \\ \vec{u} \cdot \vec{v} &= 0 \end{aligned}$$

The dot product of two vectors is 0 if and only if the two vectors are orthogonal.

Exercise 65a

Prove that $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are orthogonal in \mathbb{R} if and only if $\|\vec{u}\| = \|\vec{v}\|$.

$$\begin{aligned} (\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) &= 0 \\ \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v} &= 0 \\ \vec{u} \cdot \vec{u} &= \vec{v} \cdot \vec{v} \\ \sqrt{\vec{u} \cdot \vec{u}} &= \sqrt{\vec{v} \cdot \vec{v}} \\ \|\vec{u}\| &= \|\vec{v}\| \end{aligned}$$

Exercise 66

If $\|\vec{u}\| = 2$, $\|\vec{v}\| = \sqrt{3}$, and $\vec{u} \cdot \vec{v} = 1$, find $\|\vec{u} + \vec{v}\|$.

$$\begin{aligned} \|\vec{u} + \vec{v}\| &= \sqrt{(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})} \\ &= \sqrt{\vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}} \\ &= \sqrt{\|\vec{u}\|^2 + 2(1) + \|\vec{v}\|^2} \\ &= \sqrt{2^2 + 2 + \sqrt{3}^2} \\ &= \sqrt{4 + 2 + 3} \\ &= \sqrt{9} = 3 \end{aligned}$$

Exercise 67

Show that there are no vectors \vec{u} and \vec{v} such that $\|\vec{u}\| = 1$, $\|\vec{v}\| = 2$, and $\vec{u} \cdot \vec{v} = 3$.

$$\begin{aligned} \|\vec{u} + \vec{v}\| &\leq \|\vec{u}\| + \|\vec{v}\| \\ \sqrt{\|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2} &\leq \|\vec{u}\| + \|\vec{v}\| \\ \sqrt{1^2 + 2(3) + 2^2} &\leq 1 + 2 \\ \sqrt{1 + 6 + 4} &\leq 3 \\ \sqrt{11} &\leq 3 \\ 3.316 &\leq 3 \end{aligned}$$

Since this case violates the triangle inequality, there can be no such vectors.

Exercise 68

Prove that if \vec{u} is orthogonal to both \vec{v} and \vec{w} , then \vec{u} is orthogonal to $\vec{v} + \vec{w}$.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 0 \\ \vec{u} \cdot \vec{w} &= 0 \\ \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} &= 0 \\ \vec{u} \cdot (\vec{v} + \vec{w}) &= 0 \end{aligned}$$

Prove that if \vec{u} is orthogonal to both \vec{v} and \vec{w} , then \vec{u} is orthogonal to $s\vec{v} + t\vec{w}$ for all scalars s and t .

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 0 \\ \vec{u} \cdot \vec{w} &= 0 \\ \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} &= 0 \\ \vec{u} \cdot (\vec{v} + \vec{w}) &= 0 \\ \sum_{i=1}^n u_i(v_i + w_i) &= 0 \\ \sum_{i=1}^n u_i \sum_{i=1}^n (v_i + w_i) &= 0 \end{aligned}$$

Exercise 69

Prove that \vec{u} is orthogonal to $\vec{v} - \text{proj}_{\vec{u}}\vec{v}$ for all vectors \vec{u} and \vec{v} in \mathbb{R}^n , where $\vec{u} \neq \vec{0}$.

$$\begin{aligned} \vec{u} \cdot (\vec{v} - \text{proj}_{\vec{u}}\vec{v}) &= \vec{u} \cdot \left(\vec{v} - \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u} \right) \\ &= \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u} \cdot \vec{u} \\ &= \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} \\ &= 0 \end{aligned}$$

Exercise 70a

Prove that $\text{proj}_{\vec{u}}(\text{proj}_{\vec{u}}\vec{v}) = \text{proj}_{\vec{u}}\vec{v}$.

$$\begin{aligned}
 \vec{w} &= \text{proj}_{\vec{u}}\vec{v} \\
 &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 \text{proj}_{\vec{u}}(\text{proj}_{\vec{u}}\vec{v}) &= \text{proj}_{\vec{u}}\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 &= \text{proj}_{\vec{u}}\vec{w} \\
 &= \frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 &= \frac{\vec{u} \cdot \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\frac{\vec{u} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 &= \text{proj}_{\vec{u}}\vec{v}
 \end{aligned}$$

Exercise 70b

Prove that $\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}\vec{v}) = 0$.

$$\begin{aligned}
 \vec{w} &= \text{proj}_{\vec{u}}\vec{v} \\
 &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 \text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}\vec{v}) &= \text{proj}_{\vec{u}}(\vec{v} - \vec{w}) \\
 &= \frac{\vec{u} \cdot (\vec{v} - \vec{w})}{\vec{u} \cdot \vec{u}}\vec{u} \\
 &= \frac{\vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 &= \frac{\vec{u} \cdot \vec{v} - \vec{u} \cdot \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
 &= \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} - \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \frac{\vec{u} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u} \\
 &= \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} - \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u} \\
 &= 0\vec{u} = 0
 \end{aligned}$$

Exercise 73

Use the fact that $\text{proj}_{\vec{u}}\vec{v} = c\vec{u}$ for some scalar c together with Figure 1.41 to find c and derive the formula for $\text{proj}_{\vec{u}}\vec{v}$.

$$\begin{aligned}c\vec{u} \cdot (\vec{v} - c\vec{u}) &= 0 \\(c\vec{u} \cdot \vec{v}) - (c\vec{u} \cdot c\vec{u}) &= 0 \\c(\vec{u} \cdot \vec{v}) &= |c|^2(\vec{u} \cdot \vec{u}) \\\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} &= |c| \\\text{proj}_{\vec{u}}\vec{v} = c\vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}\end{aligned}$$

Exercise 74

Using mathematical induction, prove the following generalization of the Triangle Inequality:

$$\|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\| \leq \|\vec{v}_1\| + \|\vec{v}_2\| + \cdots + \|\vec{v}_n\|$$

Basis:

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Assumption:

$$\|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\| \leq \|\vec{v}_1\| + \|\vec{v}_2\| + \cdots + \|\vec{v}_n\|$$

Induction:

$$\begin{aligned}\|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n + \vec{v}_{n+1}\| &\leq \|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\| + \|\vec{v}_{n+1}\| \\&\leq \|\vec{v}_1\| + \|\vec{v}_2\| + \cdots + \|\vec{v}_n\| + \|\vec{v}_{n+1}\|\end{aligned}$$

Section 1.3

Exercise 1

Write the equation of the line passing through P with normal vector \vec{n} in normal form and general form.

$$\begin{aligned}P &= (0, 0) \quad \vec{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ 3x + 2y &= 0\end{aligned}$$

Exercise 3

Write the equation of the line passing through P with direction vector \vec{d} in vector form and parametric form.

$$\begin{aligned}P &= (1, 0) \quad \vec{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ l &= \begin{cases} x = 1 - t \\ y = 3t \end{cases}\end{aligned}$$

Exercise 5

Write the equation of the line passing through P with direction vector \vec{d} in vector form and parametric form.

$$P = (0, 1, 0) \quad \vec{d} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$l = \begin{cases} x = t \\ y = 1 - t \\ z = 4t \end{cases}$$

Exercise 7

Write the equation of the line passing through P with normal vector \vec{n} in normal form and general form.

$$P = (0, 1, 0) \quad \vec{n} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$3x + 2y + z = 2$$

Exercise 9

Write the equation of the plane passing through P with vectors \vec{u} and \vec{v} in vector form and parametric form.

$$P = (0, 0, 0) \quad \vec{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\Pi = \begin{cases} x = 2s - t3 \\ y = s + 2t \\ z = 2s + t \end{cases}$$

Exercise 11

Give the vector equation of the line passing through P and Q .

$$P = (1, -2) \quad Q = (3, 0)$$

$$\vec{d} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Exercise 13

Give the vector equation of the plane passing through P, Q, R .

$$P = (1, 1, 1) \quad Q = (4, 0, 2) \quad R = (0, 1, -1)$$

$$\begin{aligned}\vec{u} &= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}\end{aligned}$$

Exercise 15

Find parametric equations and an equation in vector form for the lines in \mathbb{R}^2 with the following equations:

$$y = 3x - 1$$

$$l = \begin{cases} x = t \\ y = 3t - 1 \end{cases}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Exercise 17

Suggest a “vector proof” of the fact that in \mathbb{R}^2 , two lines with slope m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

$$\vec{u} = \langle m_1, 1 \rangle$$

$$\vec{v} = \langle m_2, 1 \rangle$$

$$\vec{u} \cdot \vec{v} = 0$$

$$m_1 m_2 + 1 = 0$$

$$m_1 m_2 = -1$$

Exercise 19

The plane P_1 has the equation $4x - y + 5z = 2$. For each of the planes P in Exercise 18. determine whether P_1 and P are parallel, perpendicular, or neither.

$$\begin{aligned} \vec{n}_1 &= \langle 4, -1, 5 \rangle \\ a : 2x + 3y - z &= 1 \\ \vec{n}_a &= \langle 2, 3, -1 \rangle \\ \vec{n}_1 &\neq c\vec{n}_a \\ \vec{n}_1 \cdot \vec{n}_a &= 8 + (-3) + (-5) = 0 && \text{Perpendicular} \\ b : 4x - y + 5z &= 0 \\ \vec{n}_b &= \langle 4, -1, 5 \rangle \\ \vec{n}_1 &= c\vec{n}_a && \text{Parallel} \\ c : x - y - z &= 3 \\ \vec{n}_c &= \langle 1, -1, -1 \rangle \\ \vec{n}_1 &\neq c\vec{n}_c \\ \vec{n}_1 \cdot \vec{n}_c &= 4 + 1 + (-5) = 0 && \text{Perpendicular} \\ d : 4x + 6y - 2z &= 0 \\ \vec{n}_d &= \langle 4, 6, -2 \rangle = \langle 2, 3, -1 \rangle \\ \vec{n}_1 &\neq c\vec{n}_d \\ \vec{n}_1 \cdot \vec{n}_d &= 8 + (-3) + (-5) = 0 && \text{Perpendicular} \end{aligned}$$

Exercise 21

Find the vector form of the equation of the line in \mathbb{R}^2 that passes through $P = (2, -1)$ and is parallel to the line with general equation $2x - 3y = 1$.

$$\begin{aligned} \vec{d} &= \langle 2, 3 \rangle \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

Exercise 23

Find the vector form of the equation of the line in \mathbb{R}^3 that passes through $P = (-1, 0, 3)$ and is parallel to the line with parametric equations:

$$\begin{aligned} l &= \begin{cases} x &= 1 - t \\ y &= 2 + 3t \\ z &= -2 - t \end{cases} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \end{aligned}$$

Section 2.1

Exercise 1

Determine which equations are linear in the variables x, y, z .

$$x - \pi y + \sqrt[3]{5}z = 0$$

Linear.

Exercise 3

Determine which equations are linear in the variables x, y, z .

$$x^{-1} + 7y + z = \sin \frac{\pi}{9}$$

Nonlinear, x is not 1st degree.

Exercise 5

$$3 \cos(x) - 4y + z = \sqrt{3}$$

Nonlinear, $\cos(x)$ is a function of x .

Exercise 11

Find the solution set of each equation.

$$3x - 6y = 0$$

All points that lie on the line $y = \frac{1}{2}x$.

Exercise 13

Find the solution set of each equation.

$$x + 2y + 3z = 4$$

All points that lie on the plane $x + 2y + 3z = 4$ or $[4 - 3t - 2s, s, t]$.

Exercise 15

Determine geometrically whether each system has a unique solution, infinitely many solutions, or no solution. Then solve each system algebraically to confirm your answer.

$$x + y = 0 \quad 2x + y = 3$$

Two intersecting lines, one solution at $(3, -3)$.

Exercise 17

Determine geometrically whether each system has a unique solution, infinitely many solutions, or no solution. Then solve each system algebraically to confirm your answer.

$$3x - 6y = 3 \quad -x + 2y = 1$$

Two parallel lines, no solution.

Exercise 19

Solve the given system by back substitution.

$$\begin{aligned}x - 2y &= 1 \\y &= 3 \\x &= 7\end{aligned}$$

Exercise 21

Solve the given system by back substitution.

$$\begin{aligned}x - y + z &= 0 \\2y - z &= 1 \\3z &= -1 \\z &= -\frac{1}{3} \\y &= \frac{1}{3} \\x &= \frac{2}{3}\end{aligned}$$

Exercise 23

Solve the given system by back substitution.

$$\begin{aligned}x_1 + x_2 - x_3 - x_4 &= 1 \\x_2 + x_3 + x_4 &= 0 \\x_3 - x_4 &= 0 \\x_4 &= 1 \\x_3 &= 1 \\x_2 &= -2 \\x_1 &= 1\end{aligned}$$

Exercise 25

Solve these systems.

$$\begin{aligned}x &= 2 \\2x + y &= -3 \\-3x - 4y + z &= -1 \\y &= -7 \\z &= -23\end{aligned}$$

Exercise 27

Find the augmented matrices of the linear systems.

$$\begin{aligned}x - y &= 0 \\2x + y &= 3\end{aligned}$$
$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array} \right]$$

Exercise 29

Find the augmented matrices of the linear systems.

$$\begin{aligned}x + 5y &= -1 \\ -x + y &= -5 \\ 2x + 4y &= 4\end{aligned}$$

$$A = \left[\begin{array}{cc|c} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array} \right]$$

Exercise 31

Find a system of linear equations that has the given matrix as its augmented matrix.

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array} \right]$$

$$\begin{aligned}y + z &= 1 \\ x - y &= 1 \\ 2x - y + z &= 1\end{aligned}$$

Exercise 33

Solve the linear systems in the given equations.

$$\begin{aligned}x - y &= 0 \\ 2x + y &= 3\end{aligned}$$

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array} \right]$$

$$-2R_1 + R_2 \rightarrow R_2$$

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 3 \end{array} \right]$$

$$\frac{1}{3}R_2 \rightarrow R_2$$

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array} \right]$$

$$R_2 + R_1 \rightarrow R_1$$

$$A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right]$$

$$\begin{aligned}x &= 1 \\ y &= 1\end{aligned}$$

Exercise 35

Solve the linear systems in the given equations.

$$x + 5y = -1$$

$$-x + y = -5$$

$$2x + 4y = 4$$

$$A = \left[\begin{array}{cc|c} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array} \right]$$

$$R_1 + R_2 \rightarrow R_2$$

$$-2R_1 + R_3 \rightarrow R_3$$

$$A = \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 6 & -6 \\ 0 & -6 & 6 \end{array} \right]$$

$$\frac{1}{6}R_2 \rightarrow R_2$$

$$\frac{1}{6}R_3 \rightarrow R_3$$

$$A = \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array} \right]$$

$$5R_3 + R_1 \rightarrow R_1$$

$$A = \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array} \right]$$

$$x = 4$$

$$y = -1$$

Exercise 37

Solve the linear systems in the given equations.

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array} \right]$$

$$R_2 \leftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & 1 & 1 \end{array} \right]$$

$$-2R_1 + R_3 \rightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array} \right]$$

$$R_3 - R_2 \rightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array} \right]$$

$$0 = 2 \quad (\text{Inconsistent})$$

Exercise 39

Find a system of two linear equations in the variables x and y whose solution set is given by the parametric equations $x = t$ and $y = 3 - 2t$.

$$y = 3 - 2x$$

$$3x = 15$$

Find another parametric solution to the system in the first part in which the parameter is s and $y = s$.

$$y = s$$

$$x = \frac{s - 3}{-2}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech