Linear Algebra

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Exam 4

Problem 1a

Let x_1, x_2, x_3 be numbers. Let

$$V_3 = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

Show that $|V_3| = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

$$\begin{aligned} |V_3| &= \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} \\ &= 1(x_2x_3^2 - x_3x_2^2) - x_1(x_3^2 - x_2^2) + x_1^2(x_3 - x_2) \\ &= x_2x_3^2 - x_3x_2^2 - x_1x_3^2 + x_1x_2^2 + x_3x_1^2 - x_2x_1^2 \\ &= x_2x_3^2 - x_3x_2^2 - x_1x_3^2 + x_1x_2^2 + x_3x_1^2 - x_2x_1^2 + (x_1x_2x_3 - x_1x_2x_3) \\ &= (x_2x_3^3 - x_1x_2x_3 - x_1x_3^2 + x_1^2x_3) - (x_2^2x_3 + x_2^2x_1 + x_1x_2x_3 - x_1^2x_2) \\ &= (x_2x_3 - x_2x_1 - x_1x_3 + x_1^2)(x_3 - x_2) \\ &= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \end{aligned}$$

Problem 1b

Let $x_1, x_2, \ldots, x_n$ be numbers. Write:

$$V_n = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{bmatrix}$$

Use mathematical induction to show that:

$$|V_n| = \prod_{i<j}(x_j - x_i)$$

$$V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{vmatrix}$$

subtract row 1 from all other rows

$$= \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-2} & x_1^{n-1} \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & \ldots & x_2^{n-2} - x_1^{n-2} & x_2^{n-1} - x_1^{n-1} \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n - x_1 & x_n^2 - x_1^2 & \ldots & x_n^{n-2} - x_1^{n-2} & x_n^{n-1} - x_1^{n-1} \end{vmatrix}$$

subtract $(x_1 \times \text{column } n-1)$ from column n

subtract $(x_1 \times \text{column } n-2)$ from column $n-1$

...

subtract $(x_1 \times \text{column } 1)$ from column 2

$$= \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 - (x_1x_2 - x_1^2) & \ldots & x_2^{n-1} - x_1^{n-1} - (x_1x_2^{n-2} - x_1^{n-1}) \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n - x_1 & x_n^2 - x_1^2 - (x_1x_n - x_1^2) & \ldots & x_n^{n-1} - x_1^{n-1} - (x_1x_n^{n-2} - x_1^{n-1}) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & x_2 - x_1 & x_2^2 - x_1x_2 & \ldots & x_2^{n-1} - x_1x_2^{n-2} \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n - x_1 & x_n^2 - x_1x_n & \ldots & x_n^{n-1} - x_1x_n^{n-2} \end{vmatrix}$$

$$
\begin{aligned}
&= \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & x_2 - x_1 & x_2(x_2 - x_1) & \dots & x_2^{n-2}(x_2 - x_1) \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n - x_1 & x_n(x_n - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\
&= (x_2 - x_1) \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & x_2 & \dots & x_2^{n-2} \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n - x_1 & x_n(x_n - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\
&= (x_2 - x_1) \dots (x_n - x_1) \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & x_2 & \dots & x_2^{n-2} \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & x_n & \dots & x_n^{n-2} \end{vmatrix} \\
&= \prod_{i=2}^{n} (x_i - x_1) \begin{vmatrix} 1 & x_2 & \dots & x_2^{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \dots & x_n^{n-2} \end{vmatrix} \\
&= \prod_{i=2}^{n} (x_i - x_1) |V_{n-1}| \\
&= \prod_{i=2}^{n} (x_i - x_1) \prod_{i=3}^{n} (x_i - x_2) |V_{n-2}| \\
&= \prod_{i<j}^{n} (x_j - x_i)
\end{aligned}
$$

Problem 2

Let P be the vector space of all polynomials with real-valued coefficients. Let $D : P \to P$ be the differentiation operator. In general, $D^n : P \to P$ will be the linear transformation "take the n'th derivative".

(a) Compute the kernel of D.

$$
\begin{aligned}
D(a + bx) &= b \\
ker(D) &= \{a + bx \mid D(a + bx) = 0\} \\
&= \{a + bx \mid b = 0\} \\
&= \{a \mid a \in \mathbb{R}\}
\end{aligned}
$$

(b) Compute the kernel of D^2.

$$\begin{aligned} D^2(a+bx+cx^2) &= 2c \\ ker(D^2) &= \{a+bx+cx^2 \mid D^2(a+bx+cx^2)=0\} \\ &= \{a+bx+cx^2 \mid 2c=0\} \quad (c=0) \\ &= \{a+bx \mid a,b \in \mathbb{R}\} \end{aligned}$$

(c) Compute the kernel of D^n.

$$D^n(a_0+a_1x+a_2x^2+\cdots+a_nx^n) = na_n$$

$$\begin{aligned} ker(D^n) &= \{a_0+a_1x+a_2x^2+\cdots+a_nx^n \mid D^n(a_0+a_1x+a_2x^2+\cdots+a_nx^n)=0\} \\ &= \{a_0+a_1x+a_2x^2+\cdots+a_nx^n \mid na_n=0\} \quad (a_n=0) \\ &= \{a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1} \mid a_0,a_1,a_2,\ldots,a_{n-1} \in \mathbb{R}\} \end{aligned}$$

Problem 3

Let V be the vector space of all infinitely differentiable functions, and let $D: V \to V$ be the derivative.

(a) Let $L = D - I$, where I is the identity mapping. Compute the kernel of L.

$$ker(L) = \{a\mathrm{e}^x \mid a \in \mathbb{R}\}$$

(b) Let $L = D - aI$, where a is a number.

$$ker(L) = \{b\mathrm{e}^{ax} \mid b \in \mathbb{R}\}$$

Problem 4

(a) Let $L: V \to V$ be a linear mapping such that $L^2 = 0$. Show that $I - L$ is invertible (I is the identity mapping on V).

$$\begin{aligned} L^2 &= 0 \\ I - L^2 &= (I-L)(I+L) \\ I + 0 &= (I-L)(I+L) \\ I &= (I-L)(I+L) \\ (I-L)^{-1}I &= (I-L)^{-1}(I-L)(I+L) \\ (I-L)^{-1} &= (I+L) \end{aligned}$$

This yields the identity mapping when multiplied by $(I-L)$.

(b) Let $L : V \to V$ be a linear mapping such that $L^2 + 2L + I = 0$. Show that L is invertible.

$$\begin{aligned} L^2 + 2L + I &= 0 \\ L(L + 2I + L^{-1}I) &= 0 \\ L + 2I + L^{-1}I &= 0 \\ L^{-1} &= -L - 2I \end{aligned}$$

This yields the identity mapping when multiplied by L.

(c) Let $L : V \to V$ be a linear mapping such that $L^n = 0$, where $n \geq 3$ is an integer. Show that $I - L$ is invertible.

$$\begin{aligned} L^n &= 0 \quad (n \geq 3) \\ I - L^n &= (I - L)\left(\sum_{i=0}^{n-1} L^i\right) \\ (I - L)^{-1}(I - 0) &= (I - L)^{-1}(I - L)\left(\sum_{i=0}^{n-1} L^i\right) \\ (I - L)^{-1} &= \left(\sum_{i=0}^{n-1} L^i\right) \end{aligned}$$

This yields the identity mapping when multiplied by $I - L$.

Problem 5

Let V be a vector space. Let $P : V \to V$ be a linear map such that $P^2 = P$.

(a) Show that $V = ker(P) + range(P)$, and $ker(P) \cap range(P) = \{\vec{0}\}$.

$$\begin{aligned} P^2 &= P \\ P^2\vec{v} &= P\vec{v} \quad (\vec{v} \in V) \\ range(P) &= P(\vec{v}) \\ P^2\vec{v} - P\vec{v} &= \vec{0} \\ ker(P) &= \vec{v} - P(\vec{v}) \\ ker(P) + range(P) &= \vec{v} - P(\vec{v}) + P(\vec{v}) \\ &= \vec{v} \in V \\ &= V \end{aligned}$$

$P(P\vec{v})$ can only be zero if only $P\vec{v}$ is zero and no other element in the range can be in the kernel, therefore $ker(P) \cap range(P) = \vec{0}$ since the zero vector is the only intersection.

(b) Use the result of part (a) to conclude that V is the *direct sum* of $ker(P)$ and $range(P)$.

Because the kernel of P and the range of P only share the zero vector (whose sum is $\vec{0}$), V is the *direct sum* of the kernel and range of P.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech