Linear Algebra

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August 2017 - December 2017

Vector Spaces

Definition: Let V be a set with 2 operations called addition or multiplication. If $\vec{u}, \vec{v} \in V$, denote their sum by $\vec{u} + \vec{v}$. If c is a scalar, the **scalar multiple** of $\vec{u}$ by c is denoted $c\vec{u}$. If the following axioms hold for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all scalars c, d, then V is called a **vector space** and its elements are vectors.

1. $\vec{u} + \vec{v} \in V$ (Closure under addition)
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
4. These exists and element $\vec{0}$, called the zero vector such that $\vec{u} + \vec{0} = \vec{u}$.
5. For each vector $\vec{u} \in V$, there exists $-\vec{u} \in V$ such that $\vec{u} + (-\vec{u}) = \vec{0}$.
6. $c\vec{u} \in V$ (Closure under scalar multiplication)
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$

The scalars c, d will be taken from a field F. Usually $F = \mathbb{R}$ *or* $\mathbb{C}$. If $F = \mathbb{R}$, we say V is a real vector space. If $F = \mathbb{C}$, we say V is a complex vector space.

Examples of Vector Spaces

1. $\mathbb{R}^n$
2. M_{mn}, the set of all $m \times n$ matrices.
3. Consider the set $P_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in \mathbb{R}\}$ is a vector space where the zero polynomial is the zero vector.
4. Consider $P_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}$ is a vector space.
5. P the set of all polynomials.
6. $F = \{f : \mathbb{R} \to \mathbb{R}\}$
   - $(f + g)(x) = f(x) + g(x)$
   - $(cf)(x) = c(f(x))$
7. $F[a, b] = \{f : [a, b] \to \mathbb{R}\}$

The set of integers $\mathbb{Z}$ is not a vector space since it is not closed under multiplication. The set of complex numbers $\mathbb{C}$ is a complex vector space. $\mathbb{C}^n$ is also a complex vector space.

Theorem

Let V be a vector space with $\vec{u} \in V$ and c a scalar.

1. $0\vec{u} = \vec{0}$
2. $c\vec{0} = \vec{0}$
3. $(-1)\vec{u} = -\vec{u}$
4. $c\vec{u} = \vec{0} \to c = 0$ *or* $\vec{u} = \vec{0}$

Subspaces of V

Let V be a vector space. A subset $W \subseteq V$ is a subspace if:

1. $\vec{0} \in W$
2. $\vec{u}, \vec{u} \in W \to \vec{u} + \vec{v} \in W$
3. If $\vec{u} \in W$, then $c\vec{u} \in W$

Example

Let W be the subset of M_{nn} that are symmetric. Verify that W is a subspace.

1. Is $\vec{0} \in W$?
$$0^T = 0$$

2. Let $A, B \in W$.
$$(A+B)^T = A^T + B^T = A + B \in W$$

3. Let $A \in W$
$$(cA)^T = ca^T = cA \in W$$

Example

Let C be the set of continuous functions and let D be the set of differentiable functions. Verify that C is a subspace.

1. $\vec{0} \in C$

2. Let $f, g \in C$. Then $(f+g) \in C$ by the limit laws.

3. Let $f \in C$.
$$(cf) \in C$$

Verify that D is a subspace.

1. $\vec{0} \in D$

2. Let $f, g \in D$
$$(f+g)' = f' + g' \in D$$

3. Let $f \in D$.
$$(cf)' = cf' \in D$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech