Linear Algebra

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Determinants of Matrices

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of A is $ad - bc$.

$$\det(A) = |A| = ad - bc$$

This only works for 2×2 matrices, however. For larger matrices we have a different recursive definition. Let A be an $n \times n$ matrix. Let A_{ij} be the matrix A with row i, col j deleted. The (i, j)-cofactor of A is defined by:

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Then:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Theorem:

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} a_{ij} C_{ij} \\ &= \sum_{i=1}^{n} a_{ij} C_{ij} \end{aligned}$$

Example

Find $\det(A)$.

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\det(A) = (-1)(1)A_{21} + 0A_{22} + (-1)(2)A_{23}$$

$$\begin{aligned}\det(A) &= (-1)(1)\begin{vmatrix}-3 & 2\\ -1 & 3\end{vmatrix} + 0\begin{vmatrix}5 & 2\\ 2 & 3\end{vmatrix} + (-1)(2)\begin{vmatrix}5 & -3\\ 2 & -1\end{vmatrix}\\ &= -1(9-(-2)) + 0 + (-2)(-5-(-6))\\ &= (-1)(-7) + (-2)(1)\\ &= 7 - 2 = 5\end{aligned}$$

Some Helpful Facts

Let A be a triangular matrix (upper or lower). Then $\det(A)$ is the product of the entries on the main diagonal.

$$\det(A) = |A| = \prod_{i=1}^{n} a_{ii}$$

$$\det\left(\begin{bmatrix}2 & 3 & 4\\ 0 & 5 & 6\\ 0 & 0 & 7\end{bmatrix}\right) = 2(5)(7)$$

A is invertible if and only if $\det(A) \neq 0$.

How Determinants Are Useful

(i) The determinant of A tells you whether or not A is invertible. A is invertible if and only if $\det(A) \neq 0$.

(ii) **Cramer's Rule:** Suppose we have

$$\begin{aligned}a_{11}x_1 + \cdots + a_{1n}x_n &= b_1\\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2\\ &\vdots\\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n\end{aligned}$$

and we want to solve

$$\vec{b} = \begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$$

Let $A_i(\vec{b}) = A$ when we replace column i of A with b. If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$:

$$x_i = \frac{|A_i(\vec{b})|}{|A|}$$

Why is Cramer's Rule valid? Consider:

$$\begin{aligned}
AI_i(\vec{x}) &= A[\vec{e_1} \dots \vec{x}\vec{e_n}] \\
&= [A\vec{e_1} \dots A\vec{x} \dots A\vec{e_n}] \\
&= [\vec{a_1} \dots \vec{x} \dots \vec{a_n}] \\
&= A_i(\vec{x}) \\
|A|x_i &= |A||I_i(\vec{x})| \\
&= |AI_i(\vec{x})| \\
&= |A_i(\vec{b})| \\
x_i &= \frac{|A_i(\vec{b})|}{|A|}
\end{aligned}$$

Example

Use Cramer's Rule to solve:

$$\begin{aligned}
x_1 + 2x_2 &= 2 \\
-x_1 + 4x_2 &= 1
\end{aligned}$$

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
|A_1(\vec{b})| &= \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 2(4) - 1(2) = 6 \\
|A_2(\vec{b})| &= \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1(1) - (-1)(2) = 3 \\
|A| &= 1(4) - (-1)(2) = 6 \\
x_1 &= \frac{|A_1(\vec{b})|}{|A|} = 1 \\
x_2 &= \frac{|A_2(\vec{b})|}{|A|} = \frac{1}{2} \\
\vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

Example

Use Cramer's Rule to solve:

$$
\begin{aligned}
2x - y &= 5 \\
x + 3y &= -1
\end{aligned}
$$

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \\
|A_1(\vec{b})| &= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} = 5(3) - (-1)(-1) = 14 \\
|A_2(\vec{b})| &= \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = 2(-1) - 1(5) = -7 \\
|A| &= 2(3) - 1(-1) = 7 \\
x_1 &= \frac{|A_1(\vec{b})|}{|A|} = \frac{14}{7} = 2 \\
x_2 &= \frac{|A_2(\vec{b})|}{|A|} = \frac{-7}{7} = -1 \\
\vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}
\end{aligned}
$$

Adjoint Formula for A^{-1}

$$A^{-1} = \frac{1}{\det(A)} adjoint(A)$$

$$C_{ij} = (-1)^{i+j}(A_{ij})$$

Let $[C_{ij}]$ be the cofactor matrix of A:

$$C = [C_{ij}] \quad adjoint(A) = C^T$$

Example

Show that the eigenvalues of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are the solutions of:

$$\lambda^2 - tr(A)\lambda + \det(A) = 0$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \\ &= (a-\lambda)(d-\lambda) - bc \\ &= ad - a\lambda - d\lambda + \lambda^2 - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) \\ &= \lambda^2 - tr(A)\lambda + \det(A) = 0 \end{aligned}$$

Example

Show that the eigenvalues in the above example are:

$$\lambda = (\frac{1}{2})((a+2) \pm \sqrt{(a-d)^2 + 4bc})$$

This comes from the quadratic formula:

$$A\lambda^2 + B\lambda + C = 0$$

$$\begin{aligned} \lambda &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ A &= 1 \\ B &= -(a+d) \\ C &= ad - bc \\ \lambda &= (\frac{1}{2})((a+2) \pm \sqrt{(a-d)^2 + 4bc}) \end{aligned}$$

Example

Show that the trace and determinant of A are given by:

$$\begin{aligned}
tr(A) &= \lambda_1 + \lambda_2 \\
\det(A) &= \lambda_1\lambda_2 \\
(\lambda - \lambda_1)(\lambda - \lambda_2) &= \lambda^2 + \lambda(-\lambda_1 - \lambda_2) + \lambda_1\lambda_2 \\
&= \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1\lambda_2
\end{aligned}$$

Example

Find all values of k such that A is invertible.

$$A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$$

Find all values of k such that $|A| \neq 0$.

$$\begin{aligned}
A &= k\begin{vmatrix} 1 & -k & 3 \\ 0 & k+1 & 1 \\ 1 & -8 & k-1 \end{vmatrix} \quad \text{(factor from column)} \\
&= k\begin{vmatrix} 1 & -k & 3 \\ 0 & k+1 & 1 \\ 0 & -8+k & k-4 \end{vmatrix} \\
&= k\begin{vmatrix} k+1 & 1 \\ -8+k & k-4 \end{vmatrix} \\
&= k((k+1)(k-4) - (-8+k)) \\
&= k(k^2 - 3k - 4 + 8 - k) \\
&= k(k^2 - 4k + 4) \\
&= k(k-2)^2 \\
0 &\neq k(k-2)^2 \\
k &\neq 0, 2
\end{aligned}$$

Example

Find all values of k such that A is invertible:

$$A = \begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix}$$

$$
\begin{aligned}
|A| &= |A^T| \\
&= \begin{vmatrix} k & k^2 & 0 \\ k & 2 & k \\ 0 & k & k \end{vmatrix} \\
&= k \begin{vmatrix} 1 & k^2 & 0 \\ 1 & 2 & k \\ 0 & k & k \end{vmatrix} \\
&= k \begin{vmatrix} 1 & k^2 & 0 \\ 0 & 2-k^2 & k \\ 0 & k & k \end{vmatrix} \\
&= k \begin{vmatrix} 2-k^2 & k \\ k & k \end{vmatrix} \\
&= k^2 \begin{vmatrix} 2-k^2 & k \\ 1 & 1 \end{vmatrix} \\
&= k^2((2-k^2)-k) \\
&= k^2(-k^2-k+1) \\
&= k^2(k^2+k-2) \\
&= k^2(k+2)(k-1) \\
0 &\neq k^2(k+2)(k-1) \\
k &\neq 0, -2, 1
\end{aligned}
$$

Example

Suppose A and B are $n \times n$ matrices:

$$|A| = 3 \quad |B| = -2$$

Evaluate:

(a) $|AB|$:

$$|AB| = |A||B| = 3(-2) = 6$$

(b) $|A^2|$:

$$|A^2| = |AA| = |A||A| = (3)(3) = 9$$

(c) $|B^{-1}A|$:

$$|B^{-1}A| = |B^{-1}A| = |B^{-1}||A| = \frac{1}{|B|}|A| = \frac{1}{-2}3$$

(d) $|2A|$:

$$|2A| = 2^n|A| = (2^n)(3)$$

(e) $|3B^T|$:

$$|3B^T| = 3^n|B^T| = (3^n)|B| = 3^n(-2)$$

Example

If A is an $n \times n$ invertible matrix, show that $adjoint(A)$ is invertible.

$$\begin{aligned}
adjoint(A)^{-1} &= \frac{1}{|A|}A \\
&= adjoint(A^{-1}) \\
C &= [C_{ij}] \\
C_{ij} &= (-1)^{i+1}|A_{ij}| \\
adjoint(A) &= C^T \\
A^{-1} &= \frac{1}{|A|}adjoint(A) \\
\therefore |A|A^{-1} &= adjoint(A)
\end{aligned}$$

Example

If A is an $n \times n$ matrix, then show:

$$|adjoint(A)| = |A|^{n-1}$$

$$\left||A|A^{-1}\right| = |A|^n|A^{-1}| = |A|^n(\frac{1}{|A|}) = |A|^{n-1}$$

Example

Show that, for any square matrix A, A and A^T have the same characteristic polynomial.

$$(A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$$

The characteristic polynomial of $A = |A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I|$.

Example

Let A be a nilpotent matrix. Show that $\lambda = 0$ is the only eigenvalue of A. Show that $\lambda = 0$ is a legitimate eigenvalue.

$$|A^m| = |0| = 0$$

Thus $\lambda = 0$ is an eigenvalue. Show that there are no other eigenvalues.
Suppose λ is another eigenvalue ($\lambda \neq 0$). Then λ^m is an eigenvalue of A^m. This forces $\lambda^m = 0 \therefore \lambda = 0$.

Example

Let A be an idempotent matrix. Show that $\lambda = 0, \lambda = 1$ are the only eigenvalues.
Since A is idempotent, it means that $A^2 = A$. Let λ be an eigenvalue of A, then λ^2 is an eigenvalue of $A^2 = A$.

$$\begin{aligned} \lambda^2 &= \lambda \\ \lambda^2 - \lambda &= 0 \\ \lambda(\lambda - 1) &= 0 \\ \lambda &= 0 \quad \lambda = 1 \end{aligned}$$

Example

If $\vec{v}$ is an eigenvector of A with eigenvalue λ and c is a scalar, show that $\vec{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

$$(A - cI)(\vec{v}) = A\vec{v} - (cI)(\vec{v}) = \lambda\vec{v} - c\vec{v} = (\lambda - c)\vec{v}$$

Companion Matrix

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The companion matrix of p is defined by:

$$C(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Example

Find the companion matrix of $p(x) = x^2 - 7x + 12$.

$$C(p) = \begin{bmatrix} 7 & -12 \\ 1 & 0 \end{bmatrix}$$

Now find the characteristic polynomial for $C(p)$:

$$\begin{aligned} |C(p) - \lambda I| &= \begin{vmatrix} 7-\lambda & -12 \\ 1 & -\lambda \end{vmatrix} \\ &= (-\lambda)(7-\lambda) - 1(-12) \\ &= \lambda^2 - 7\lambda + 12 \\ &= (-1)^2 p(\lambda) \end{aligned}$$

Example

Find the companion matrix of $p(x) = x^3 + 3x^2 - 4x + 12$.

$$C(p) = \begin{bmatrix} -3 & 4 & 12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is the characteristic polynomial for $C(p)$? We find the characteristic polynomial for $C(p)$ by taking:

$$\begin{aligned} |C(p) - \lambda I| &= -p(\lambda) \\ &= (-1)^3 p(\lambda) \end{aligned}$$

In general, if $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, the characteristic polynomial of $C(p) = (-1)^n p(\lambda)$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech