Linear Algebra

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Eigenvectors and Eigenvalues

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there exists a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$. Such a vector $\vec{x}$ is called an **eigenvector** of A corresponding to λ.
In $\mathbb{R}^2$, $A\vec{x} = \lambda x$ means that the action of A on $\vec{x}$ just yields a vector parallel to $\vec{x}$.

Properties

- $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$.
- $(A - \lambda I)\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$.
- $null(A - \lambda I)$ is nontrivial.
- $\det(A - \lambda I) = 0$ (allows us to solve for λ to find eigenvalues).
- $|A - \lambda I| = 0$ is called the characteristic polynomial.

We define $E_\lambda = null(A - \lambda I)$ to be the eigenspace corresponding to λ. Eigenvalues are the roots of $|A - \lambda I|$ and eigenspaces $E_{\lambda_i} = null(A - \lambda_i I)$.

Example

Show that $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 4\lambda \end{aligned}$$

So $\lambda = 4$ is an eigenvalue for A and $\vec{x}$ is a corresponding eigenvector.

Example

Show there exists a non-zero vector $\vec{x}$ satisfying $A\vec{x} = 5\vec{x}$.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Conclude: $\lambda = 5$ is an eigenvalue of A.

$$\begin{aligned} A\vec{x} &= 5\vec{x} \\ (A - 5I)\vec{x} &= \vec{0} \\ A - 5I &= \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix} \\ x_1 - \frac{1}{2}x_2 &= 0 \\ x_1 &= \frac{1}{2}x_2 \\ \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ E_5 &= span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \end{aligned}$$

Example

Show that $\lambda = 6$ is an eigenvalue of:

$$A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

We need to show $(A - 6I)\vec{x} = 6$ has a non-trivial solution.

$$A - 6I = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
x_1 + x_2 - 2x_3 &= 0 \\
x_1 &= -x_2 + 2x_3 \\
\vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \\
E_6 &= span \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right)
\end{aligned}$$

Example

Find the eigenvectors and eigenvalues over $\mathbb{R}$ and $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}, i = \sqrt{-1}$ of the following:

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\begin{aligned}
A - \lambda I &= \begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \\
(1-\lambda)(-1-\lambda) &= 0 \\
1 - \lambda &= 0 \\
-1 - \lambda &= 0 \\
\lambda &= \pm 1
\end{aligned}$$

For $\lambda = 1$:

$$\begin{aligned}
A - I &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
\vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
E_1 &= span \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)
\end{aligned}$$

For $\lambda = -1$:

$$A + I = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$E_1 = span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

(ii) $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$B - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$
$$(3-\lambda)(3-\lambda) - 1 = 0$$
$$9 - 6\lambda + \lambda^2 - 1 = 0$$
$$(\lambda - 4)(\lambda - 2) = 0$$
$$\lambda = 4 \quad \lambda = 2$$

For $\lambda = 2$:

$$B - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$x_1 + x_2 = 0$$
$$x_1 = -x_2$$
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$E_2 = span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

For $\lambda = 4$:

$$
\begin{aligned}
B - 4I &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \\
x_1 - x_2 &= 0 \\
x_1 &= x_2 \\
\vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
E_4 &= span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)
\end{aligned}
$$

(iii) $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$
\begin{aligned}
C - \lambda I &= \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \\
(-\lambda)^2 - (1)(-1) &= 0 \\
\lambda^2 + 1 &= 0 \\
\lambda^2 &= -1 \\
\lambda &= \pm i
\end{aligned}
$$

Depending on our domain of interest, we have no real-valued eigenvalues and complex eigenvalues $\pm i$.

Theorems

Theorem 1. The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof:

$$
\begin{aligned}
A &= \begin{bmatrix} a_{11} & \dots & \dots & \dots \\ 0 & a_{22} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & a_{nn} \end{bmatrix} \\
0 &= |A - \lambda I| \\
&= \begin{vmatrix} a_{11} - \lambda & \dots & \dots & \dots \\ 0 & a_{22} - \lambda & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & a_{nn} - \lambda \end{vmatrix} \\
&= (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda) \\
\lambda &= a_{11}, a_{22}, \dots, a_{nn}
\end{aligned}
$$

Theorem 2. A square matrix A is invertible if and only if 0 is not an eigenvalue of A.

Proof:

$$
\begin{aligned}
A \text{ is invertible} &\leftrightarrow |A| \neq 0 \\
&\leftrightarrow |A - 0I| \neq 0 \\
&\leftrightarrow \lambda = 0 \text{ is not an eigenvalue of } A
\end{aligned}
$$

Theorem 3. Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\vec{x}$, then:

(i) For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\vec{x}$.

Proof:

Base Case $n = 1$:

$$A^1\vec{x} = \lambda^1\vec{x}$$

Induction Hypothesis:

$$A^n\vec{x} = \lambda^n\vec{x}$$

Induction Step:

$$\begin{aligned} A^{n+1}\vec{x} &= A(A^n\vec{x}) \\ &= A(\lambda^n\vec{x}) \\ &= \lambda^n(A\vec{x}) \\ &= \lambda^n(\lambda\vec{x}) \\ &= \lambda^{n+1}\vec{x} \end{aligned}$$

(ii) If A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$.
Proof:

$$\begin{aligned} \lambda(A^{-1}\vec{x}) &= A^{-1}(\lambda\vec{x}) \\ &= A^{-1}(A\vec{x}) \\ &= I\vec{x} \\ &= \vec{x} \end{aligned}$$

$$\therefore \lambda(A^{-1}\vec{x}) = \vec{x} \leftrightarrow A^{-1}\vec{x} = (\frac{1}{\lambda})\vec{x}$$

(iii) If A is invertible, the for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\vec{x}$.
Proof:
We only need to show this for negative integers.
Base Case $n = -1$: Because of part (ii), this is true.
Induction Hypothesis:

$$A^{-n}\vec{x} = \frac{1}{\lambda^n}\vec{x}$$

Induction Step: Assume it is true for $-n$ and show it is true for $-n-1$.

$$\begin{aligned} A^{-(n+1)}\vec{x} &= A^{-1}(A^{-n}\vec{x}) \\ &= A^{-1}(\frac{1}{\lambda^n}\vec{x}) \\ &= (\frac{1}{\lambda^n})(A^{-1}\vec{x}) \\ &= (\frac{1}{\lambda^n})(\frac{1}{\lambda}\vec{x}) \\ &= (\frac{1}{\lambda^{n+1}})\vec{x} \end{aligned}$$

Theorem 4. Suppose A is $n \times n$, and A has eigenvectors $\vec{v_1}, \ldots, \vec{v_m}$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$. If $\vec{x} \in \mathbb{R}^n$ is $\vec{x} = c_1\vec{v_1} + \cdots + c_m\vec{v_m}$, then for any integer k:

$$A^k\vec{x} = c_1\lambda_1^k\vec{v_1} + \cdots + c_m\lambda_m^k\vec{v_m}$$

Proof:

$$\begin{aligned} \vec{x} &= c_1\vec{v_1} + \cdots + c_m\vec{v_m} \\ A^k\vec{x} &= A^k(c_1\vec{v_1} + \cdots + c_m\vec{v_m}) \\ &= c_1A^k\vec{v_1} + \cdots + c_mA^k\vec{v_m} \\ &= c_1\lambda_1^k\vec{v_1} + \cdots + c_m\lambda_m^k\vec{v_m} \end{aligned}$$

Theorem 5. Let A be an $n \times n$ matrix. Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of A. Let $\vec{v_1}, \ldots, \vec{v_m}$ be the corresponding eigenvectors. Then $\vec{v_1}, \ldots, \vec{v_m}$ are linearly independent.

Proof (by contradiction):

Suppose $\vec{v_1}, \ldots, \vec{v_m}$ are linearly dependent. Choose the smallest index $k+1$ such that $\overrightarrow{v_{k+1}} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$. Note that by the minimality of $k+1$, $\vec{v_1}, \ldots, \vec{v_k}$ are linearly dependent.

$$\begin{aligned} A\overrightarrow{v_{k+1}} &= A(c_1\vec{v_1} + \cdots + c_k\vec{v_k}) \\ \lambda_{k+1}\overrightarrow{v_{k+1}} &= c_1\lambda_1\vec{v_1} + \cdots + c_k\lambda_k\vec{v_k} \\ \lambda_{k+1}\overrightarrow{v_{k+1}} &= c_1\lambda_{k+1}\vec{v_1} + \cdots + c_k\lambda_{k+1}\vec{v_k} \\ \vec{0} &= c_1(\lambda_1 - \lambda_{k+1})\vec{v_1} + \cdots + c_k(\lambda_k - \lambda_{k+1})\vec{v_k} \end{aligned}$$

By the linear independence of $\vec{v_1}, \ldots, \vec{v_k}$ the scalars are zero, this implies that $c_1 = c_2 = -c_k = 0$ and thus $\overrightarrow{v_{k+1}} = \vec{0}$. Since eigenvectors cannot be 0, this is a contradiction and thus our initial assumption must be false.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech