

Linear Algebra

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Eigenvectors and Eigenvalues

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there exists a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$. Such a vector \vec{x} is called an **eigenvector** of A corresponding to λ .

In \mathbb{R}^2 , $A\vec{x} = \lambda\vec{x}$ means that the action of A on \vec{x} just yields a vector parallel to \vec{x} .

Properties

- $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$.
- $(A - \lambda I)\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$.
- $\text{null}(A - \lambda I)$ is nontrivial.
- $\det(A - \lambda I) = 0$ (allows us to solve for λ to find eigenvalues).
- $|A - \lambda I| = 0$ is called the characteristic polynomial.

We define $E_\lambda = \text{null}(A - \lambda I)$ to be the eigenspace corresponding to λ . Eigenvalues are the roots of $|A - \lambda I|$ and eigenspaces $E_{\lambda_i} = \text{null}(A - \lambda_i I)$.

Example

Show that $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 4\lambda \end{aligned}$$

So $\lambda = 4$ is an eigenvalue for A and \vec{x} is a corresponding eigenvector.

Example

Show there exists a non-zero vector \vec{x} satisfying $A\vec{x} = 5\vec{x}$.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Conclude: $\lambda = 5$ is an eigenvalue of A .

$$\begin{aligned} A\vec{x} &= 5\vec{x} \\ (A - 5I)\vec{x} &= \vec{0} \\ A - 5I &= \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix} \\ x_1 - \frac{1}{2}x_2 &= 0 \\ x_1 &= \frac{1}{2}x_2 \\ \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ E_5 &= \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \end{aligned}$$

Example

Show that $\lambda = 6$ is an eigenvalue of:

$$A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

We need to show $(A - 6I)\vec{x} = \vec{0}$ has a non-trivial solution.

$$A - 6I = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 - 2x_3 = 0$$

$$x_1 = -x_2 + 2x_3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$E_6 = \text{span} \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Example

Find the eigenvectors and eigenvalues over \mathbb{R} and $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}, i = \sqrt{-1}$ of the following:

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{bmatrix}$$

$$(1 - \lambda)(-1 - \lambda) = 0$$

$$1 - \lambda = 0$$

$$-1 - \lambda = 0$$

$$\lambda = \pm 1$$

For $\lambda = 1$:

$$A - I = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$E_1 = \text{span} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

For $\lambda = -1$:

$$\begin{aligned}A + I &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ E_1 &= \text{span} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)\end{aligned}$$

(ii) $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$\begin{aligned}B - \lambda I &= \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix} \\ (3 - \lambda)(3 - \lambda) - 1 &= 0 \\ 9 - 6\lambda + \lambda^2 - 1 &= 0 \\ (\lambda - 4)(\lambda - 2) &= 0 \\ \lambda = 4 \quad \lambda = 2\end{aligned}$$

For $\lambda = 2$:

$$\begin{aligned}B - 2I &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ x_1 + x_2 &= 0 \\ x_1 &= -x_2 \\ \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ E_2 &= \text{span} \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)\end{aligned}$$

For $\lambda = 4$:

$$B - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_4 = \text{span} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$(iii) \ C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$C - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$(-\lambda)^2 - (1)(-1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$

Depending on our domain of interest, we have no real-valued eigenvalues and complex eigenvalues $\pm i$.

Theorems

Theorem 1. The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof:

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & \dots & \dots & \dots \\ 0 & a_{22} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & a_{nn} \end{bmatrix} \\ 0 &= |A - \lambda I| \\ &= \begin{vmatrix} a_{11} - \lambda & \dots & \dots & \dots \\ 0 & a_{22} - \lambda & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & a_{nn} - \lambda \end{vmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda) \\ \lambda &= a_{11}, a_{22}, \dots, a_{nn} \end{aligned}$$

Theorem 2. A square matrix A is invertible if and only if 0 is not an eigenvalue of A .

Proof:

$$\begin{aligned} A \text{ is invertible} &\leftrightarrow |A| \neq 0 \\ &\leftrightarrow |A - 0I| \neq 0 \\ &\leftrightarrow \lambda = 0 \text{ is not an eigenvalue of } A \end{aligned}$$

Theorem 3. Let A be a square matrix with eigenvalue λ and corresponding eigenvector \vec{x} , then:

- (i) For any positive integer n , λ^n is an eigenvalue of A^n with corresponding eigenvector \vec{x} .

Proof:

Base Case $n = 1$:

$$A^1 \vec{x} = \lambda^1 \vec{x}$$

Induction Hypothesis:

$$A^n \vec{x} = \lambda^n \vec{x}$$

Induction Step:

$$\begin{aligned}A^{n+1}\vec{x} &= A(A^n\vec{x}) \\ &= A(\lambda^n\vec{x}) \\ &= \lambda^n(A\vec{x}) \\ &= \lambda^n(\lambda\vec{x}) \\ &= \lambda^{n+1}\vec{x}\end{aligned}$$

- (ii) If A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector \vec{x} .

Proof:

$$\begin{aligned}\lambda(A^{-1}\vec{x}) &= A^{-1}(\lambda\vec{x}) \\ &= A^{-1}(A\vec{x}) \\ &= I\vec{x} \\ &= \vec{x}\end{aligned}$$

$$\therefore \lambda(A^{-1}\vec{x}) = \vec{x} \leftrightarrow A^{-1}\vec{x} = \left(\frac{1}{\lambda}\right)\vec{x}$$

- (iii) If A is invertible, then for any integer n , λ^n is an eigenvalue of A^n with corresponding eigenvector \vec{x} .

Proof:

We only need to show this for negative integers.

Base Case $n = -1$: Because of part (ii), this is true.

Induction Hypothesis:

$$A^{-n}\vec{x} = \frac{1}{\lambda^n}\vec{x}$$

Induction Step: Assume it is true for $-n$ and show it is true for $-n - 1$.

$$\begin{aligned}A^{-(n+1)}\vec{x} &= A^{-1}(A^{-n}\vec{x}) \\ &= A^{-1}\left(\frac{1}{\lambda^n}\vec{x}\right) \\ &= \left(\frac{1}{\lambda^n}\right)(A^{-1}\vec{x}) \\ &= \left(\frac{1}{\lambda^n}\right)\left(\frac{1}{\lambda}\vec{x}\right) \\ &= \left(\frac{1}{\lambda^{n+1}}\right)\vec{x}\end{aligned}$$

Theorem 4. Suppose A is $n \times n$, and A has eigenvectors $\vec{v}_1, \dots, \vec{v}_m$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_m$. If $\vec{x} \in \mathbb{R}^n$ is $\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$, then for any integer k :

$$A^k \vec{x} = c_1 \lambda_1^k \vec{v}_1 + \dots + c_m \lambda_m^k \vec{v}_m$$

Proof:

$$\begin{aligned} \vec{x} &= c_1 \vec{v}_1 + \dots + c_m \vec{v}_m \\ A^k \vec{x} &= A^k (c_1 \vec{v}_1 + \dots + c_m \vec{v}_m) \\ &= c_1 A^k \vec{v}_1 + \dots + c_m A^k \vec{v}_m \\ &= c_1 \lambda_1^k \vec{v}_1 + \dots + c_m \lambda_m^k \vec{v}_m \end{aligned}$$

Theorem 5. Let A be an $n \times n$ matrix. Let $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of A . Let $\vec{v}_1, \dots, \vec{v}_m$ be the corresponding eigenvectors. Then $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.

Proof (by contradiction):

Suppose $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent. Choose the smallest index $k + 1$ such that $\vec{v}_{k+1} = c_1 \vec{v}_1 + \dots + c_k \vec{v}_k$. Note that by the minimality of $k + 1$, $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent.

$$\begin{aligned} A \vec{v}_{k+1} &= A(c_1 \vec{v}_1 + \dots + c_k \vec{v}_k) \\ \lambda_{k+1} \vec{v}_{k+1} &= c_1 \lambda_1 \vec{v}_1 + \dots + c_k \lambda_k \vec{v}_k \\ \lambda_{k+1} \vec{v}_{k+1} &= c_1 \lambda_{k+1} \vec{v}_1 + \dots + c_k \lambda_{k+1} \vec{v}_k \\ \vec{0} &= c_1 (\lambda_1 - \lambda_{k+1}) \vec{v}_1 + \dots + c_k (\lambda_k - \lambda_{k+1}) \vec{v}_k \end{aligned}$$

By the linear independence of $\vec{v}_1, \dots, \vec{v}_k$ the scalars are zero, this implies that $c_1 = c_2 = \dots = c_k = 0$ and thus $\vec{v}_{k+1} = \vec{0}$. Since eigenvectors cannot be 0, this is a contradiction and thus our initial assumption must be false.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech