

# Linear Algebra

Alvin Lin

August 2017 - December 2017

## Linear Transformations

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a transformation if for each  $\vec{x} \in \mathbb{R}^n \exists! T(\vec{x}) \in \mathbb{R}^m$ .  $Range(T) = Image(T) = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$ .  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is one-to-one if  $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$ ,

$$T(\vec{x}) = T(\vec{y}) \longrightarrow \vec{x} = \vec{y}$$

$$\vec{x} \neq \vec{y} \longrightarrow T(\vec{x}) \neq T(\vec{y})$$

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is **onto** (surjective) if  $Range(T) = Codomain(T)$ .

## Properties of Linear Transformations

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a transformation.  $T$  is a linear transformation if for all  $\vec{u}, \vec{v} \in \mathbb{R}^n$  and all scalars  $c, d$ :

1.  $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2.  $T(c\vec{u}) = cT(\vec{u})$
3.  $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

### Example

Suppose  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by:

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 3y \\ -4x \end{bmatrix}$$

Verify that  $T$  is linear. Let:

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 3(y_1 + y_2) \\ -4(x_1 + x_2) \end{bmatrix} \\ &= \begin{bmatrix} 3y_1 + 3y_2 \\ -4x_1 - 4x_2 \end{bmatrix} \\ &= \begin{bmatrix} 3y_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} 3y_2 \\ -4x_2 \end{bmatrix} \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

$$\begin{aligned} T(c\vec{u}) &= \begin{bmatrix} cx \\ cy \end{bmatrix} \\ &= \begin{bmatrix} 3(cy) \\ -4(cx) \end{bmatrix} \\ &= c \begin{bmatrix} 3y \\ -4x \end{bmatrix} \\ &= cT(\vec{u}) \end{aligned}$$

$T$  satisfies both properties of a linear transformation. To prove a transformation is not linear, one only needs to find a single counterexample.

### Example

Consider  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  as a linear transformation. What is  $T(\vec{0})$ ?

$$\begin{aligned} T(\vec{0}) &= T(\vec{0} + \vec{0}) \\ &= T(\vec{0}) + T(\vec{0}) \\ \vec{0} &= T(\vec{0}) \end{aligned}$$

## Facts about Linear Transformations

Suppose  $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  basis for  $\mathbb{R}^n$ . Let  $\vec{x} \in \mathbb{R}^n$ .

$$\begin{aligned}\vec{x} &= \sum_{i=1}^n x_i \vec{v}_i \\ T(\vec{x}) &= T\left(\sum_{i=1}^n x_i \vec{v}_i\right) \\ &= \sum_{i=1}^n x_i T(\vec{v}_i)\end{aligned}$$

Let the standard matrix  $A_T$  stand for the linear transformation  $T$ :

$$\begin{aligned}A_T &= [A] \\ &= [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \dots \quad T(\vec{e}_n)]\end{aligned}$$

Let  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $T : \mathbb{R}^m \rightarrow \mathbb{R}^k$ . The composition of linear transformations  $T \circ S : \mathbb{R}^n \rightarrow \mathbb{R}^k$  is also linear.

$$\begin{aligned}[T] &= B \\ [S] &= A \\ (T \circ S)(\vec{x}) &= T(S(\vec{x})) \\ &= T(A\vec{x}) \\ &= B(A\vec{x}) \\ &= (BA)\vec{x}\end{aligned}$$

### Example

Define  $\Pi_x : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ :

$$\Pi_x \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$\Pi_x$  is linear. Find  $[\Pi_x]$ :

$$\begin{aligned}\Pi_x(\vec{e}_1) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \Pi_x(\vec{e}_2) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ A = [\Pi_x] &= [\Pi_x(\vec{e}_1) \quad \Pi_x(\vec{e}_2)] \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

## Theorems and Definitions

Let  $T : V \rightarrow W$  be linear.  $\ker(T)$  is the set of all inputs where the output through the transformation  $T$  is 0. Then:

1.  $\ker(T)$  is a subspace of  $V$ . Proof:

(a)  $T(\vec{0}) = \vec{0}$ . Thus  $\vec{0} \in \ker(T)$ .

(b) Assume  $\vec{u}, \vec{v} \in \ker(T)$ .

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{0} + \vec{0} = \vec{0}$$

Thus  $\vec{u} + \vec{v} \in \ker(T)$ .

(c) Assume  $\vec{u} \in \ker(T)$  and  $c$  is a scalar.

$$T(c\vec{u}) = cT(\vec{u}) = c\vec{0} = \vec{0}$$

2.  $\text{Range}(T)$  is a subspace of  $W$ . Proof:

(a)  $T(\vec{0}) = \vec{0}$ . So  $\vec{0} \in \text{range}(T)$ .

(b) Let  $\vec{u}, \vec{v} \in \text{range}(T)$ . Then there exists  $\vec{\alpha}_u, \vec{\alpha}_v$  such that  $T(\vec{\alpha}_u) = \vec{u}$  and  $T(\vec{\alpha}_v) = \vec{v}$ .

$$\vec{u} + \vec{v} = T(\vec{\alpha}_u) + T(\vec{\alpha}_v) = T(\vec{\alpha}_u + \vec{\alpha}_v)$$

So  $\vec{u} + \vec{v} \in \text{range}(T)$ .

(c) Let  $\vec{u} \in \text{range}(T)$ . There exists  $\vec{\alpha}_u \in V$  such that  $T(\vec{\alpha}_u) = \vec{u}$ .

$$T(c\vec{\alpha}_u) = cT(\vec{\alpha}_u) = c\vec{u}$$

Thus  $c\vec{u} \in \text{range}(T)$ .

**Definition**

Let  $T : V \rightarrow W$  be linear. The rank of  $T$  is:

$$\dim(\text{range}(T)) = \text{rank}(T)$$

**Definition**

The nullity of  $T$  is:

$$\dim(\text{ker}(T)) = \text{nullity}(T)$$

**The Rank Theorem**

If  $T : V \rightarrow W$  is linear, then:

$$\dim(V) = \text{rank}(T) + \text{nullity}(T)$$

**Theorem**

Let  $\dim(V) = \dim(W)$ . Then a linear  $T : V \rightarrow W$  is one-to-one if and only if it is onto.

**Theorem**

Let  $T : V \rightarrow W$  be linear. If  $S = \{\vec{v}_1, \dots, \vec{v}_2\}$  is linearly independent, then  $T(S) = \{T(\vec{v}_1), \dots, T(\vec{v}_2)\}$  is linearly independent in  $W$ . Let  $\dim(V) = \dim(W) = n$ . Then a one-to-one linear transformation  $T : V \rightarrow W$  maps a basis for  $V$  to a basis for  $W$ .

**Theorem**

A linear transformation  $T : V \rightarrow W$  is invertible if and only if it is one-to-one and onto.

### Example

Find the kernel and the range of the differential operator  $D : P_3 \rightarrow P_2$  defined by  $D(p(x)) = p'(x)$ .

$$\ker(D) = \{p(x) \in P_3 \mid D(p(x)) = 0\}$$

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2 = 0$$

$$a_1 = a_2 = a_3 = 0$$

$$\ker(D) = \{a_0 \mid a_0 \in \mathbb{R}\}$$

$$\text{Basis for } \ker(D) = \{1\}$$

$$\text{nullity}(D) = 1$$

$$4 = \dim(P_3) = \text{rank}(D) + \text{nullity}(D)$$

$$\text{rank}(D) = 3$$

$$\text{range}(D) = P_2$$

$D$  is onto.

### Example

Define  $S : P_1 \rightarrow \mathbb{R}$  by:  $S(p(x)) = \int_0^1 p(x) dx$ . Find the kernel and range of the transformation.

$$S(p(x)) = 0$$

$$S(a + bx) = 0$$

$$= \int_0^1 a + bx dx = \left[ ax + \frac{bx^2}{2} \right]_0^1 = a + \frac{b}{2} = 0$$

$$a = -\frac{b}{2}$$

$$\ker(D) = \left\{ -\frac{b}{2} + bx \mid b \in \mathbb{R} \right\}$$

$$\text{Basis for } \ker(D) = \left\{ -\frac{1}{2} + x \right\}$$

$$\text{nullity}(D) = 1$$

$$2 = \dim(P_1) = \text{rank}(D) + \text{nullity}(D)$$

$$\text{rank}(D) = 1 = \dim(\mathbb{R})$$

$$\text{range}(D) = \mathbb{R}$$

$D$  is onto.

## Isomorphism

Let  $V, W$  be vector spaces. We say  $T : V \rightarrow W$  is an isomorphism if  $T$  is invertible. We say  $V$  is isomorphic to  $W$ . We denote this as  $V \cong W$ . Suppose  $V, W$  have finite dimension. Then  $V \cong W$  if and only if  $V$  and  $W$  have the same dimension. Example: Is  $\mathbb{R}^4$  isomorphic to  $M_{22}$ ?

$$\begin{aligned} \dim(\mathbb{R}^4) &= 4 \\ B &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \in M_{22} \\ \dim(M_{22}) &= 4 \\ \mathbb{R}^4 &\cong M_{22} \end{aligned}$$

### Example

Is  $P_3$  isomorphic to  $\mathbb{R}^3$ ?

$$\dim(P_3) = 4 \neq \dim(\mathbb{R}^3) = 3$$

### Example

Suppose  $V, W$  are both  $n$ -dimensional. Compute the number of isomorphisms  $T : V \rightarrow W$ .

$$\begin{aligned} \text{Basis for } V &= \{\vec{v}_1, \dots, \vec{v}_n\} \\ \text{Basis for } W &= \{\vec{w}_1, \dots, \vec{w}_n\} \end{aligned}$$

Number of isomorphisms:  $n!$

## Matrix Associated to a Linear Transformation

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$A = [T(\vec{e}_1) \mid T(\vec{e}_2) \mid \dots \mid T(\vec{e}_n)]$$

$$T : V \rightarrow W \quad \dim(V) = n \quad \dim(W) = m$$

$V$  has basis  $B = \{\vec{v}_1, \dots, \vec{v}_n\}$  and  $W$  has basis  $G = \{\vec{w}_1, \dots, \vec{w}_n\}$ .

$$\begin{array}{ccc}
 V & \xrightarrow{T} & W \\
 S \downarrow & & \downarrow S \\
 \mathbb{R}^n & \xrightarrow{S \circ T \circ R^{-1}} & \mathbb{R}^m
 \end{array}$$

**Example**

The transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is linear, defined by:

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$

The basis for  $\mathbb{R}^3 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = B$ . The basis for  $\mathbb{R}^2 = \{\vec{e}_2, \vec{e}_1\} = G$ . Calculate  $T$ .

**Example**

The transformation  $T : P_2 \rightarrow P_3$  is defined by:

$$T(p(x)) = p(2x - 1)$$

The basis for  $P_2 = \mathbb{E} = \{1, x, x^2\}$ . Calculate  $[T]_{\mathbb{E} \leftarrow \mathbb{E}} = [T]_{\mathbb{E}}$ .

$$T(1) = 1$$

$$T(x) = 2x - 1$$

$$T(x^2) = (2x - 1)^2 = 4x^2 - 4x + 1$$

$$[T(1)]_{\mathbb{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(x)]_{\mathbb{E}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$[T(x^2)]_{\mathbb{E}} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

$$[T]_{\mathbb{E}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$



You can find all my notes at <http://omgimanagerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanagerd.tech](mailto:alvin@omgimanagerd.tech)