Linear Algebra

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Linear Transformations

$T : \mathbb{R}^n \to \mathbb{R}^m$ is a transformation if for each $\vec{x} \in \mathbb{R}^n$ $\exists!$ $T(\vec{x}) \in \mathbb{R}^m$. $Range(T) = Image(T) = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$. $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$,

$$T(\vec{x}) = T(\vec{y}) \longrightarrow \vec{x} = \vec{u}$$

$$\vec{x} \neq \vec{y} \longrightarrow T(\vec{x}) \neq \vec{y}$$

$T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** (surjective) if $Range(T) = Codomain(T)$.

Properties of Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation. T is a linear transformation if for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and all scalars c, d:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(c\vec{u}) = cT(\vec{u})$
3. $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{u})$

Example

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3y \\ -4x \end{bmatrix}$$

Verify that T is linear. Let:

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\begin{aligned}
T(\vec{u}+\vec{v}) &= T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 3(y_1 + y_2) \\ -4(x_1 + x_2) \end{bmatrix} \\
&= \begin{bmatrix} 3y_1 + 3y_2 \\ -4x_1 - 4x_2 \end{bmatrix} \\
&= \begin{bmatrix} 3y_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} 3y_2 \\ -4x_2 \end{bmatrix} \\
&= T(\vec{u}) + T(\vec{v}) \\
T(c\vec{u}) &= \left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) \\
&= \begin{bmatrix} 3(cy) \\ -4(cx) \end{bmatrix} \\
&= c\begin{bmatrix} 3y \\ -4x \end{bmatrix} \\
&= cT(\vec{u})
\end{aligned}$$

T satisfies both properties of a linear transformation. To prove a transformation is not linear, one only needs to find a single counterexample.

Example

Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ as a linear transformation. What is $T(\vec{0})$?

$$\begin{aligned}
T(\vec{0}) &= T(\vec{0} + \vec{0}) \\
&= T(\vec{0}) + T(\vec{0}) \\
\vec{0} &= T(\vec{0})
\end{aligned}$$

Facts about Linear Transformations

Suppose $B = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ basis for R^n. Let $\vec{x} \in \mathbb{R}^n$.

$$
\begin{aligned}
\vec{x} &= \sum_{i=1}^{n} x_1 \vec{v_1} \\
T(\vec{x}) &= T\left(\sum_{i=1}^{n} x_i \vec{v_i}\right) \\
&= \sum_{i=1}^{n} x_i T(\vec{v_i})
\end{aligned}
$$

Let the standard matrix A_T stand for the linear transformation T:

$$
\begin{aligned}
A_T &= [A] \\
&= \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \ldots & T(\vec{e_n}) \end{bmatrix}
\end{aligned}
$$

Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^k$. The composition of linear transformations $T \circ S : \mathbb{R}^n \to \mathbb{R}^k$ is also linear.

$$
\begin{aligned}
[T] &= B \\
[S] &= A \\
(T \circ S)(\vec{x}) &= T(S(\vec{x})) \\
&= T(A\vec{x}) \\
&= B(A\vec{x}) \\
&= (BA)\vec{x}
\end{aligned}
$$

Example

Define $\Pi_x : \mathbb{R}^2 \to \mathbb{R}^2$:

$$
\Pi_x\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}
$$

Π_x is linear. Find $[\Pi_x]$:

$$\Pi_x(\vec{e_1}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\Pi_x(\vec{e_2}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$A = [\Pi_x] = \begin{bmatrix} \Pi_x(\vec{e_1}) & \Pi_x(\vec{e_2}) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Theorems and Definitions

Let $T : V \to W$ be linear. $ker(T)$ is the set of all inputs where the output through the transformation T is 0. Then:

1. $ker(T)$ is a subspace of V. Proof:
   (a) $T(\vec{0}) = \vec{0}$. Thus $\vec{0} \in ker(T)$.
   (b) Assume $\vec{u}, \vec{v} \in ker(T)$.
   $$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{V}) = \vec{0} + \vec{0} = \vec{0}$$
   Thus $\vec{u} + \vec{v} \in ker(T)$.
   (c) Assume $\vec{u} \in ker(T)$ and c is a scalar.
   $$T(c\vec{u}) = cT(\vec{u}) = c\vec{0} = \vec{0}$$
2. $Range(T)$ is a subspace of W. Proof:
   (a) $T(\vec{0}) = \vec{0}$. So $\vec{0} \in range(T)$.
   (b) Let $\vec{u}, \vec{v} \in range(T)$. Then there exists $\vec{\alpha_u}, \vec{\alpha_v}$ such that $T(\vec{\alpha_u}) = \vec{u}$ and $T(\vec{\alpha_v}) = \vec{v}$.
   $$\vec{u} + \vec{v} = T(\vec{\alpha_u}) + T(\vec{\alpha_v}) = T(\vec{\alpha_u} + \vec{\alpha_v})$$
   So $\vec{u} + \vec{v} \in range(T)$.
   (c) Let $\vec{u} \in range(T)$. There exists $\vec{\alpha_u} \in V$ such that $T(\vec{\alpha_u}) = \vec{u}$.
   $$T(c\vec{\alpha_u}) = cT(\vec{\alpha_u}) = c\vec{u}$$
   Thus $c\vec{u} \in range(T)$.

Definition

Let $T : V \to W$ be linear. The rank of T is:

$$dim(range(T)) = rank(T)$$

Definition

The nullity of T is:

$$dim(ker(T)) = nullity(T)$$

The Rank Theorem

If $T : V \to W$ is linear, then:

$$dim(V) = rank(T) + nullity(T)$$

Theorem

Let $dim(V) = dim(W)$. Then a linear $T : V \to W$ is one-to-one if and only if it is onto.

Theorem

Let $T : V \to W$ be linear. If $S = \{\vec{v_1}, \ldots, \vec{v_2}\}$ is linearly independent, then $T(S) = \{T(\vec{v_1}), \ldots, T(\vec{v_2})\}$ is linearly independent in W. Let $dim(V) = dim(W) = n$. Then a one-to-one linear transformation $T : V \to W$ maps a basis for V to a basis for W.

Theorem

A linear transformation $T : V \to W$ is invertible if and only if it is one-to-one and onto.

Example

Find the kernel and the range of the differential operator $D : P_3 \to P_2$ defined by $D(p(x)) = p'(x)$.

$$\begin{aligned}
ker(D) &= \{p(x) \in P_3 \mid D(p(x)) = 0\} \\
D(a_0 + a_1x + a_2x^2 + a_3x^3) &= a_1 + 2a_2x + 3a_3x^2 = 0 \\
a_1 &= a_2 = a_3 = 0 \\
ker(D) &= \{a_0 \mid a_0 \in \mathbb{R}\} \\
\text{Basis for } ker(D) &= \{1\} \\
nullity(D) &= 1 \\
4 &= dim(P_3) = rank(D) + nullity(D) \\
rank(D) &= 3 \\
range(D) &= P_2
\end{aligned}$$

D is onto.

Example

Define $S : P_1 \to R$ by: $S(p(x)) = \int_0^1 p(x)\ \mathrm{d}x$. Find the kernel and range of the transformation.

$$\begin{aligned}
S(p(x)) &= 0 \\
S(a + bx) &= 0 \\
&= \int_0^1 a + bx\ \mathrm{d}x = \left[ax + \frac{bx^2}{2}\right]_0^1 = a + \frac{b}{2} = 0 \\
a &= \frac{-b}{2} \\
ker(D) &= \{-\frac{b}{2} + bx \mid b \in \mathbb{R}\} \\
\text{Basis for } ker(D) &= \{-\frac{1}{2} + x\} \\
nullity(D) &= 1 \\
2 &= dim(P_1) = rank(D) + nullity(D) \\
rank(D) &= 1 = dim(\mathbb{R}) \\
range(D) &= \mathbb{R}
\end{aligned}$$

D is onto.

Isomorphism

Let V, W be vector spaces. We say $T : V \to W$ is an isomorphism if T is invertible. We say V is isomorphic to W. We denote this as $V \cong W$. Suppose V, W have finite dimension. Then $V \cong W$ if and only if V and W have the same dimension. Example: Is R^4 isomorphic to M_{22}?

$$\begin{aligned} dim(\mathbb{R}^4) &= 4 \\ B &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \in M_{22} \\ dim(M_{22}) &= 4 \\ R^4 &\cong M_{22} \end{aligned}$$

Example

Is P_3 isomorphic to R^3?

$$dim(P_3) = 4 \neq dim(R^3) = 3$$

Example

Suppose V, W are both n-dimensional. Compute the number of isomorphisms $T : V \to W$.

$$\text{Basis for } V = \{\vec{v_1}, \ldots, \vec{v_n}\}$$
$$\text{Basis for } W = \{\vec{w_1}, \ldots, \vec{w_n}\}$$

Number of isomorphisms: $n!$

Matrix Associated to a Linear Transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$A = [T(\vec{e_1}) \mid T(\vec{e_2}) \mid \cdots \mid T(\vec{e_n})]$$

$$T : V \to W \quad dim(V) = n \quad dim(W) = m$$

V has basis $B = \{\vec{v_1}, \ldots, \vec{v_n}\}$ and W has basis $G = \{\vec{w_1}, \ldots, \vec{w_n}\}$.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "S"'] & W \arrow[d, "S"] \\ \mathbb{R}^n \arrow[r, "S \circ T \circ R^{-1}"'] & \mathbb{R}^m \end{tikzcd}$$

Example

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear, defined by:

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$

The basis for $\mathbb{R}^3 = \{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = B$. The basis for $\mathbb{R}^2 = \{\vec{e_2}, \vec{e_1}\} = G$. Calculate T.

Example

The transformation $T : P_2 \to P_3$ is defined by:

$$T(p(x)) = p(2x - 1)$$

The basis for $P_2 = \mathbb{E} = \{1, x, x^2\}$. Calculate $[T]_{\mathbb{E} \leftarrow \mathbb{E}} = [T]_{\mathbb{E}}$.

$$T(1) = 1$$
$$T(x) = 2x - 1$$
$$T(x^2) = (2x - 1)^2 = 4x^2 - 4x + 1$$
$$[T(1)]_{\mathbb{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$[T(x)]_{\mathbb{E}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$
$$[T(x^2)]_{\mathbb{E}} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$
$$[T]_{\mathbb{E}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech