Linear Algebra

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Matrices

A **matrix** is a rectangular array of numbers. Example:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{7} & 1 & 0 \\ 2 & 3\pi & \frac{3}{4} \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ 17 \end{bmatrix} \text{(column matrix)}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \text{(row matrix)}$$

An $m \times n$ matrix has m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Columns of A are denoted as: $\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}$
Rows of A are denoted as: $\vec{A_1}, \vec{A_2}, \dots, \vec{A_m}$.

$$A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \dots & \vec{a_n} \end{bmatrix} = \begin{bmatrix} \vec{A_1} \\ \vec{A_2} \\ \vdots \\ \vec{A_m} \end{bmatrix}$$

For any matrix A, a_{ij} is the entry in row i and column j.

Special Matrices

1. **Square Matrix**: A matrix with the same number of row and columns, an $n \times n$ matrix.

2. **Diagonal Matrix**: A square matrix where every entry off the diagonal is 0.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3. **Scalar Matrix**: A diagonal matrix where all entries on the diagonal are equal.

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

4. **Identity Matrix**: A square matrix where $a_{ii} = 1$ for $1 \leq i \leq n$ is 1.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. **Zero Matrix**: The matrix where all entries are 0.

Matrix Equality

In order for matricies A and B to be equal, the sizes of A and B must agree, and all of the corresponding entries are equal.

$$
\begin{aligned}
A &= \begin{bmatrix} 5 & 0 & 2 \\ 4 & 1 & -2 \end{bmatrix} \\
B &= \begin{bmatrix} 5 & 0 & 2 \\ x & y & z \end{bmatrix} \\
x &= 4 \quad y = 1 \quad z = -2 \\
B^T &= \begin{bmatrix} 5 & x \\ 0 & y \\ 2 & z \end{bmatrix} \\
B^T &\neq B \\
B^T &\neq A \\
\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} &\neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}
$$

Addition of Matrices

$$A = [a_{ij}] \quad B = [b_{ij}]$$

If A and B have the same sizes, let $S = A + B$ where:

$$s_{ij} = a_{ij} + b_{ij}$$

Additive Identity: Let $0_{m\times n}$ (or 0) denote the zero matrix:

1. $A + 0 = A = 0 + A$
2. $A + (-B) = A - B$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$$

$A + B$ is not defined in this case.

$$C = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$A + C = \begin{bmatrix} 0 & 6 \\ 5 & 7 \end{bmatrix}$$

Scalar Multiplication

Let c be a scalar and A be a matrix:

$$cA = [ca_{ij}]$$

Example

$$3 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 12 & 15 \end{bmatrix}$$

Matrix Multiplication

Let A be an $m \times n$ matrix and B be an $n \times r$ matrix. $A \times B = P$ is an $m \times r$ matrix derived from the following:

$$P_{ij} = \vec{A_i} \cdot \vec{b_j} = \sum_{k=1}^{n} a_{ij} b_{kj}$$

$$A \times B = A \begin{bmatrix} \vec{b_1} & \vec{b_2} & \dots & \vec{b_r} \end{bmatrix} = \begin{bmatrix} A\vec{b_1} & A\vec{b_2} & \dots A\vec{b_r} \end{bmatrix}$$

$$A \times B = \begin{bmatrix} \vec{A_1} \\ \vec{A_2} \\ \vdots \\ \vec{A_m} \end{bmatrix} B = \begin{bmatrix} \vec{A_1}B \\ \vec{A_2}B \\ \vdots \\ \vec{A_m}B \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$
$$B = \begin{bmatrix} 4 & 1 \\ -1 & 3 \end{bmatrix}$$
$$A \times B = \begin{bmatrix} 1(4) + (-2)(-1) & 1(1) + (-2)(3) \\ 3(4) + (2)(-1) & (-2)(-1) + (2)(3) \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 10 & 9 \end{bmatrix}$$

Example

Consider the linear system:

$$\begin{aligned} 2x - 3y + 5z &= 7 \\ -3x + 4y - 6z &= 2 \\ 4x + y - z &= 0 \end{aligned}$$

As a matrix equation:

$$\begin{bmatrix} 2 & -3 & 5 \\ -3 & 4 & -6 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix}$$
$$A\vec{x} = \vec{b}$$

Recall the "standard basis vectors" $\vec{e_i}$ where there is a 1 in the ith place and 0's elsewhere. Let $\vec{e_i}$ be the ith standard column vector and $\vec{e_j}$ be the standard row vector.

$$\begin{aligned} A\vec{e_i} &= \vec{a_i} \\ \vec{e_j} &= \vec{A_j} \end{aligned}$$

Power of Matrices

Let A be a square matrix, look at:

$$A^k = A \times A \times \ldots A$$

By convention:

$$\begin{aligned} A^0 &= I \\ A^1 &= A \\ A^k &= A^{k-1}A \end{aligned}$$

Example

Let:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Then:

$$\begin{aligned}
A^2 &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\
A^3 = A^2A &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \\
A^4 = A^3A &= \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \\
A^n &= \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}
\end{aligned}$$

Proof by Induction:

- Base Case:

$$A^1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^0 & 2^0 \\ 2^0 & 2^0 \end{bmatrix}$$

- Induction Step: Assume it is true for n, prove:

$$\begin{aligned}
A^{n+1} &= \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix} \\
A^{n+1} &= A^nA \\
&= \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2^{n-1}2^{n-1} & 2^{n-1}2^{n-1} \\ 2^{n-1}2^{n-1} & 2^{n-1}2^{n-1} \end{bmatrix} \\
&= \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}
\end{aligned}$$

Transpose of a Matrix

A^T will denote the **transpose** of A. The row i, column j entry of $A^T = a_{ji}$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

If A is not square: $A^T \neq A$
Recall that the dot product $\vec{u} \cdot \vec{v}$ is equal to:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \sum_{i=1}^{n} u_i v_i \\ &= \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\ &= \vec{u^T} \vec{v} \end{aligned}$$

Symmetric Matrices

A square matrix A is **symmetric** if $A = A^T$. Example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$I_2^T = I_2$, therefore I_2 is symmetric. I_n is also symmetric.

Properties of Matrices

Let A, B, C be matrices of the same size. Let c, d be scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$

4. $A + (-A) = 0$
5. $c(A + B) = cA + cB$
6. $(c + d)A = cA + dA$
7. $c(dA) = (cd)A$
8. $1A = A$

Properties of matrix multiplication (k is a scalar):

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $k(AB) = (kA)B = A(kB)$
5. $IA = AI = A$

Properties of transpose:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = k(A^T)$
4. $(AB)^T = B^T A^T$
5. $(A^n)^T = (A^T)^n$

Theorems:

1. If A is a square matrix, $A + A^T$ is symmetric.
2. For any matrix A, AA^T and $A^T A$ are symmetric.

Extension:

1. $(A_1 + A_2 + \cdots + A_n)^T = \sum_{i=1}^{n} (A_i)^T$
2. $(A_1 \cdot A_2 \cdot \cdots \cdot A_n)^T = (A_n)^T (A_{n-1})^T (A_1)^T$

Example

Let $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Suppose $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in span(A_1, A_2, A_3)$.

$$c_1A_1 + c_2A_2 + c_3A_3 = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$
$$\begin{bmatrix} c_2 + c_3 & c_1 + c_3 \\ -c_1 + c_3 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} c_2 + c_3 \qquad = w$$
$$c_1 + c_3 = x$$
$$-c_1 + c_3 = y$$
$$c_2 + c_3 = z$$
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & w \\ 1 & 0 & 1 & x \\ -1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & x - \frac{x+y}{2} \\ 0 & 1 & 0 & w - \frac{x+y}{2} \\ 0 & 0 & 1 & \frac{x+y}{2} \\ 0 & 0 & 0 & z - w \end{array}\right]$$

This describes the general form of all matrixes in the span.

Inverse of a Matrix

Assume A is a square matrix. If we try to solve $A\vec{x} = \vec{b}$, we would like to cancel A out. We want a matrix A' such that $A'A = AA' = I$.

$$A\vec{x} = \vec{b}$$
$$A'(A\vec{x}) = A'\vec{b}$$
$$(A'A)\vec{x} = A'\vec{b}$$
$$I\vec{x} = A'\vec{b}$$
$$\vec{x} = A'\vec{v}$$

When this A' does exist, we say A is invertible.
Theorem: If A is an invertible $n \times n$ matrix, then the system of linear equations given by $A\vec{x} = \vec{b}$ has a unique solution.
Let A be a 2×2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Suppose $ad - bc \neq 0$, then A is invertible:

$$\begin{aligned} A' &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

The inverse of a matrix is also denoted as A^{-1}.

General Procedure for Finding A^{-1}

$$[A|I_n] \to [I|A^{-1}]$$

By reducing a matrix to reduced row echelon form, it is possible to find the inverse of A. This is possible when $rank(A) < n$.
Fact:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Properties of Invertible Matrices

Suppose A is invertible.

1. $(A^{-1})^{-1} = A$
2. $(cA)^{-1} = \frac{1}{c}A^{-1}$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$
5. $(A^n)^{-1} = (A^{-1})^n$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech