Linear Algebra

Alvin Lin

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Spanning Sets and Linear Independence

We want to know when one vector is a **linear combination** of some other vectors.

$$c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

For example, is the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ a linear combination of $\vec{v} = \begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}-1\\1\\-3\end{bmatrix}$.
The solution would be to find scalars c_1, c_2, c_3 such that:

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = c_1\begin{bmatrix}1\\0\\3\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\-3\end{bmatrix}$$

We can treat this as a system of linear equations and solve an augmented matrix:

$$A' = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right]$$

Bring this to reduced row echelon form yields:

$$A' = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is a linear combination of $\vec{v}, \vec{w}$ through the constants:

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

Describing a Linear System

There are three equivalent ways to specify a linear system:

1. $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{cases}$$

2. Let $\vec{A_i}$ be a column vector with entries a_i:
$$x_1\vec{A_1} + x_2\vec{A_2} + \cdots + x_n\vec{A_n} = \vec{b}$$

3. $A\vec{x} = \vec{b}$ where A is an $m \times x$ matrix:
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$
where we multiply $A\vec{x} = \vec{b}$ using:
$$b_i = \text{ith row of A} \times \vec{x}$$

Theorem: A system of linear equations with augmented matrix $[A|\vec{b}]$ is consistent if and only if $\vec{b}$ is a linear combination of the columns of A.

Spanning Sets

Let $\{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of vectors.

$$Span(\{\vec{v_1}, \ldots, \vec{v_k}\}) = \left\{ \sum_{i=1}^{k} c_i\vec{v_i} \mid c_i\text{'s are scalars} \right\}$$

The span of:

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is equal to:

$$\begin{aligned} Span(\{\vec{e_1}, \vec{e_2}\}) &= \{c_1\vec{e_1} + c_2\vec{e_2} \mid c_1, c_2 \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_2, c_2 \in \mathbb{R} \right\} \\ &= \mathbb{R}^2 \end{aligned}$$

Is a vector $\vec{v}$ in $Span(\{\vec{u}, \vec{w}\})$? Can we find scalars c_1, c_2 such that $\vec{v} = c_1\vec{u} + c_2\vec{w}$? This translates to the following:

$$A = \begin{bmatrix} \vec{u} & \vec{w} \end{bmatrix} \quad \vec{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

And we are essentially solving a system of linear equations:

$$A' = [A \mid \vec{v}]$$

Linear Independence

We say vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ are **linearly independent** if:

$$\sum_{i=1}^{n} c_i\vec{v_i} = \vec{0} \Rightarrow \forall i(c_i = 0)$$

In terms of matrices, let $A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \ldots \vec{v_m} \end{bmatrix}$. Examining $A\vec{x} = \vec{0}$, the only solution for $\vec{x}$ is:

$$\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

A set of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ is **linearly dependent** if it is not linearly independent.

Example

Are the vectors $\vec{0}, \vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ linearly dependent or indepedent?

$$c_1(\vec{0}) + c_2\vec{v_1} + c_3\vec{v_2} + \cdots + c_{m+1}\vec{v_m} = \vec{0}$$

c_1 can be 1, so this set of vectors is linearly dependent.

Example

Are the vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

linearly independent or linearly dependent?

$$c_1\vec{v} + c_2\vec{w} \stackrel{?}{=} \vec{0}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This has a unique solution, so $\vec{v}, \vec{w}$ are linearly independent.

Example

The general equation of the plane contains points (1,0,3), (-1,1,-3), and the origin. It has equation $ax + by + cz = 0$. Find a, b, c.

$$\begin{aligned} a(1) + 3c &= 0 \\ a(-1) + 1b - 3c &= 0 \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right] &= \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \\ a + 3c &= 0 \\ a &= -3c \\ b &= 0 \\ -3cx + cz &= 0 \\ -3x + z &= 0 \end{aligned}$$

Example

Prove that $\vec{u}, \vec{v}, \vec{w} \in span(\vec{u}, \vec{v}, \vec{w})$.

$$\begin{aligned}\vec{u} &= 1\vec{u} + 0\vec{v} + 0\vec{w} \\ \vec{v} &= 0\vec{u} + 1\vec{v} + 0\vec{w} \\ \vec{w} &= 0\vec{u} + 0\vec{v} + 1\vec{w} \\ \vec{u}, \vec{v}, \vec{w} &\in span(\vec{u}, \vec{v}, \vec{w})\end{aligned}$$

Example

Prove that $\vec{u}, \vec{v}, \vec{w} \in span(\vec{u}, \vec{u}+\vec{v}, \vec{u}+\vec{v}+\vec{w})$.

$$\begin{aligned}\vec{u} &= 1\vec{u} + 0(\vec{u}+\vec{v}) + 0(\vec{u}+\vec{v}+\vec{w}) \\ \vec{v} &= -1\vec{u} + 1(\vec{u}+\vec{v}) + 0(\vec{u}+\vec{v}+\vec{w}) \\ \vec{w} &= 0\vec{u} - 1(\vec{u}+\vec{v}) + 1(\vec{u}+\vec{v}+\vec{w}) \\ \vec{u}, \vec{v}, \vec{w} &\in span(\vec{u}, \vec{u}+\vec{v}, \vec{u}+\vec{v}+\vec{w})\end{aligned}$$

Useful Fact

Suppose we have m vectors in $\mathbb{R}^n$, where $m > n$. Those vectors are linearly dependent.

$$A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \dots & \vec{v_m} \end{bmatrix}$$

Examine: $A\vec{x} = \vec{0}$. As a consequence of the rank theorem, which states that the number of free variables is equal to the number of columns of A minus the rank of A, $A\vec{x} = \vec{0}$ has a non-trivial solution, hence the columns of A are linearly dependent.

Example

Prove that if $\vec{u_1}, \dots, \vec{u_m} \in \mathbb{R}^n$ where $S = \{\vec{u_1}, \dots, \vec{u_k}\}$ and $T = \{\vec{u_1}, \dots, \vec{u_k}, \vec{u_{k+1}}, \dots, \vec{u_m}\}$ that $span(S) \subseteq span(T)$. Let $\alpha \in span(S)$:

$$\begin{aligned}\alpha &= c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_k\vec{u_k} \\ &= c_1\vec{u_1} + \cdots + c_k\vec{u_k} + 0\vec{u_{k+1}} + 0\vec{u_{k+2}} + 0\vec{u_m} \\ &\in span(T)\end{aligned}$$

Deduce if $\mathbb{R}^n = span(S)$, then $\mathbb{R}^n = span(T)$.

$$\mathbb{R}^n = span(S) \subseteq span(T) \subseteq \mathbb{R}^n$$

$$span(T) = \mathbb{R}^n$$

Example

Suppose $\vec{w}$ is a linear combination of $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}$ and each $\vec{u_1}$ is a linear combination of $\vec{v_1}, \ldots, \vec{v_m}$. Prove $\vec{w}$ is a linear combination of $\vec{v_1}, \ldots, \vec{v_m}$.

$$\begin{aligned}
\vec{w} &= \sum_{i=1}^{k} c_i u_i \\
&= \sum_{i=1}^{k} c_i \left(\sum_{j=1}^{m} d_{ij} \vec{v_j} \right) \\
&= \sum_{i=1}^{k} \left(\sum_{j=1}^{m} c_i d_{ij} \vec{v_j} \right) \\
&= \sum_{j=1}^{m} \left(\sum_{i=1}^{k} c_i d_{ij} \right) \vec{v_i} \\
w &\in span(\vec{v_1}, \ldots, \vec{v_m})
\end{aligned}$$

Also suppose each $\vec{v_j}$ is a linear combination of $\vec{u_1}, \ldots, \vec{u_k}$. Prove $span(\vec{u_1}, \ldots, \vec{u_k}) = span(\vec{v_1}, \ldots, \vec{v_m})$. Above we proved:

$$span(\vec{u_1}, \ldots, \vec{u_k}) \subseteq span(\vec{v_1}, \ldots, \vec{v_m})$$

Therefore:

$$span(\vec{u_1}, \ldots, \vec{u_k}) \supseteq span(\vec{v_1}, \ldots, \vec{v_m})$$

Example

If the columns of an $n \times n$ matrix A are linearly independent as vectors in $\mathbb{R}^n$, what is the rank of A?

$$rank(A) = n$$

The columns are linearly independent as $A\vec{x} = \vec{0}$ has only the trivial solution. So there are no free variables.

Example

Prove two vectors are linearly dependent if and only if 1 is a scalar multiple of the other.

Say $\vec{u}$ and $\vec{v}$ are linearly depedent, there exists scalars c_1, c_2 (not both 0) such that $c_1\vec{u} + c_2\vec{v} = \vec{0}$. Without loss of generality, we say that $c_1 \neq 0$, then:

$$\begin{aligned} c_1\vec{u} &= -c_2\vec{v} \\ \vec{u} &= \frac{-c_2}{c_1}\vec{v} \end{aligned}$$

Suppose $\vec{u}$ is a scalar multiple of $\vec{v}$.

$$\begin{aligned} \vec{u} &= c\vec{v} \\ c\vec{v} - \vec{u} &= \vec{0} \end{aligned}$$

Thus, $\vec{u}$ and $\vec{v}$ are linearly dependent.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech