Linear Algebra

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Linear Algebra

The study of linear algebra is about two basic things. We study **vector spaces** and structure preserving maps between vector spaces. A **vector space** is set v with two binary operations, addition and scalar multiplcation. A vector space will satisfy various distributive identities.

$$\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$$

are all vector spaces.

Vectors

A **vector** is a directed line segment corresponding to the displacement from points A to B (in $\mathbb{R}^2$). If a vector starts at the origin, we will say that the vector is in standard position. We can represent them as row vectors or column vectors depending on convenience. Each is an ordered pair.

$$\vec{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The zero vector ($\vec{0}$) is a special vector.

$$\vec{0} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Let $\vec{v} \in \mathbb{R}^2$:

$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$$

Adding Vectors in $\mathbb{R}^2$

Take the following:

$$\vec{v} = \langle v_1, v_2 \rangle$$
$$\vec{w} = \langle w_1, w_2 \rangle$$
$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

Example:

$$\vec{v} = \langle 1, 2 \rangle$$
$$\vec{w} = \langle 3, 4 \rangle$$
$$\vec{v} + \vec{w} = \langle 1 + 3, 2 + 4 \rangle = \langle 4, 6 \rangle$$

Scalar Multiplication

Let c be a scalar and let $\vec{v} = \langle v_1, v_2 \in \mathbb{R}^2$.

$$c \cdot \vec{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle$$

Example:

$$2 \cdot \langle 3, 4 \rangle = \langle 2(3), 2(4) \rangle = \langle 6, 8 \rangle$$

If $c > 0$, then $c \cdot \vec{v}$ points in the same direction as $\vec{v}$.
If $c < 0$, then $c \cdot \vec{v}$ points in the opposite direction as $\vec{v}$.
If $|c| > 1$, then $c \cdot \vec{v}$ is $\vec{v}$ stretched by a factor of c.
If $|c| < 1$, then $c \cdot \vec{v}$ is $\vec{v}$ compressed by a factor of c.

Vectors in $\mathbb{R}^3$

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

Similarly, vectors in $\mathbb{R}^n$ are:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid all\ x_i \in \mathbb{R}\}$$

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$, let $c \in \mathbb{R}$:

$$\vec{u} + \vec{v} = \langle u_1, u_2, \ldots, u_n \rangle + \langle v_1, v_2, \ldots, v_n \rangle + \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle$$
$$c\vec{v} = c\langle v_1, v_2, \ldots, v_n \rangle = \langle cv_1, cv_2, \ldots, cv_n \rangle$$

Algebraic Properties of $\mathbb{R}^n$

Let:

$$\begin{aligned}\vec{u} &= \langle u_1, u_2, \ldots, u_n \rangle \in \mathbb{R}^n \\ \vec{v} &= \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{R}^n \\ \vec{w} &= \langle w_1, w_2, \ldots, w_n \rangle \in \mathbb{R}^n\end{aligned}$$

Let $c, d \in \mathbb{R}$ (scalars):

1. Commutative
$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$
2. Associative
$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$
3.
$$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$$
4.
$$\vec{u} + (-\vec{u}) = \vec{0}$$
5. Distributive
$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$$
6.
$$(c + d)\vec{u} = c\vec{u} + d\vec{u}$$
7.
$$c(d\vec{u}) = (cd)\vec{u}$$
8.
$$1\vec{u} = \vec{u}$$

Proof of (1):

$$\begin{aligned}\vec{u} + \vec{v} &= \langle u_1, u_2, \ldots, u_n \rangle + \langle v_1, v_2, \ldots, v_n \rangle \\ &= \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle \\ &= \langle v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n \rangle \\ &= \langle v_1, v_2, \ldots, v_n \rangle + \langle u_1, u_2, \ldots, u_n \rangle \\ &= \vec{v} + \vec{u}\end{aligned}$$

Proof of (2):

$$
\begin{aligned}
(\vec{u}+\vec{v})+\vec{w} &= (\langle u_1,\ldots,u_n\rangle + \langle v_1,\ldots,v_n\rangle) + \langle w_1,\ldots,w_n\rangle \\
&= \langle u_1+v_1,\ldots,u_n+v_n\rangle + \langle w_1,\ldots,w_n\rangle \\
&= \langle (u_1+v_1)+w_1,\ldots,(u_n+v_n)+w_n\rangle \\
&= \langle u_1,\ldots,u_n\rangle + (\langle v_1+w_1,\ldots,v_n+w_n) \\
&= \vec{u} + (\langle v_1,\ldots,v_n\rangle + \langle w_1,\ldots,w_n\rangle) \\
&= \vec{u} + (\vec{v}+\vec{w})
\end{aligned}
$$

Proof of (3):

$$
\begin{aligned}
\vec{u}+\vec{0} &= \langle u_1,u_2,\ldots,u_n\rangle + \langle 0,0,\ldots,0\rangle \\
&= \langle u_1+0,u_2+0,\ldots,u_n+0\rangle \\
&= \langle u_1,u_2,\ldots,u_n\rangle = \vec{u}
\end{aligned}
$$

Proof of (4):

$$
\begin{aligned}
\vec{u}+(-\vec{u}) &= \langle u_1,u_2,\ldots,u_n\rangle + \langle -u_1,-u_2,\ldots,-u_n\rangle \\
&= \langle u_1-u_1,u_2-u_2,\ldots,u_n-u_n\rangle \\
&= \langle 0,0,\ldots,0\rangle = \vec{0}
\end{aligned}
$$

Proof of (5):

$$
\begin{aligned}
c(\vec{u}+\vec{v}) &= c(\langle u_1,u_2,\ldots,u_n\rangle + \langle v_1,v_2,\ldots,v_n\rangle) \\
&= c(\langle u_1+v_1,u_2+v_2,\ldots,u_n+v_n\rangle) \\
&= \langle c(u_1+v_1),c(u_2+v_2),\ldots,c(u_n+v_n)\rangle \\
&= \langle cu_1+cv_1,cu_2+cv_2,\ldots,cu_n+cv_n\rangle \\
&= \langle cu_1,cu_2,\ldots,cu_n\rangle + \langle cv_1,cv_2,\ldots,cv_n\rangle \\
&= c\langle u_1,u_2,\ldots,u_n\rangle + c\langle v_1,v_2,\ldots,v_n\rangle \\
&= c\vec{u}+c\vec{v}
\end{aligned}
$$

Linear Combinations

Let $\vec{v_1},\vec{v_2},\ldots,\vec{v_k} \in \mathbb{R}^n$. We say $\vec{v}$ is a **linear combination** of $\vec{v_1},\ldots,\vec{v_k}$ if there exists scalars $c_1,\ldots,c_k$ such that:

$$\vec{v} = \sum_{i=1}^{k} c_i\vec{v_i} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$$

Example in $\mathbb{R}^3$: Let:

$$\vec{v} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \quad \vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
$$\vec{v_2} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$$

Claim: $\vec{v}$ is a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$:
We must find c_1, c_2, c_3 such that:

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

Dot Product

Let:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$
$$\begin{aligned} \vec{u} \cdot \vec{v} &= \sum_{i=1}^{n} u_i v_i \\ &= u_1v_1 + u_2v_2 + \cdots + u_nv_n \end{aligned}$$

Rules for the Dot Product

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, c is a scalar:

1. $$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

2. $$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

3.
$$(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$$

4.
$$\vec{u} \cdot \vec{u} \geq 0$$
$$\vec{u} \cdot \vec{u} = 0 \text{ iff } \vec{u} = \vec{0}$$

Proof for (1):

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \sum_{i=1}^{n} u_i v_i \\ &= \sum_{i=1}^{n} v_i u_i \\ &= \vec{v} \cdot \vec{u}\end{aligned}$$

Proof for (2):

$$\begin{aligned}\vec{u} \cdot (\vec{v} + \vec{w}) &= \vec{u} \cdot (\langle v_1 + w_1, \ldots, v_n + w_n \rangle) \\ &= \sum_{i=1}^{n} u_i (v_i + w_i) \\ &= \sum_{i=1}^{n} (u_i v_i + u_i w_i) \\ &= \sum_{i=1}^{n} u_i v_i + \sum_{i=1}^{n} u_i w_i \\ &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}\end{aligned}$$

Proof for (3):

$$\begin{aligned}\sum_{i=1}^{n} (c u_i) v_i &= \sum_{i=1}^{n} c (u_i v_i) \\ &= c \sum_{i=1}^{n} u_i v_i \\ &= c(\vec{u} \cdot \vec{v})\end{aligned}$$

Proof for (4):

$$\begin{aligned}\vec{u} \cdot \vec{u} &= \sum_{i=1}^{n} u_i u_i \\ &= \sum_{i=1}^{n} (u_i)^2\end{aligned}$$

$(u_1)^2$ is non-negative, therefore the summation must be greater than or equal to 0.

Norm or Length of a Vector

Let:

$$\vec{v} \in \mathbb{R}^n \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{\vec{v} \cdot \vec{v}} \\ &= \sqrt{\sum_{i=1}^{n} (v_1)^2}\end{aligned}$$

Example

In $\mathbb{R}^2$, consider $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Calculate $|\vec{v}|$.

$$\|\vec{v}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Properties of a vector norm

$$|\vec{v}| = 0 \iff \vec{v} = \vec{0}$$

This is true because $\|\vec{v}\| = 0 \iff \vec{v} \cdot \vec{v} = 0 \iff \vec{v} = \vec{0}$.

$$\|c\vec{v}\| = |c|\|\vec{v}\|$$

Proof:

$$\begin{aligned}
\|c\vec{v}\|^2 &= (c\vec{v}) \cdot (c\vec{v}) \\
&= c^2(\vec{v} \cdot \vec{v}) \\
&= c^2\|\vec{v}\|^2 \\
\sqrt{\|c\vec{v}\|^2} &= \sqrt{c^2\|\vec{v}\|^2} \\
\|\vec{v}\| = |c|\|\vec{v}\|
\end{aligned}$$

Unit Vectors

A **unit vector** is a vector of length 1. In $\mathbb{R}^2$:

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In $\mathbb{R}^3$:

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In $\mathbb{R}^n$, there exist the unit vectors $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$. e_i has a 1 in the i^th component and zeros everywhere else.

Vector Normalization

In $\mathbb{R}^3$ with $\vec{v} \in \mathbb{R}^n$ and $\vec{v} \neq \vec{0}$, the unit vector corresponding to $\vec{v}$ is:

$$\vec{v}_{norm} = \frac{1}{\|\vec{v}\|}\vec{v}$$

Example

Normalize $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$:

$$\begin{aligned} \|\vec{v}\| &= \sqrt{2^2 + (-1)^2 + (-3)^2} \\ &= \sqrt{4+1+9} \\ &= \sqrt{14} \\ \vec{v}_{norm} &= \frac{1}{\|\vec{v}\|}\vec{v} \\ &= \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{bmatrix} \end{aligned}$$

Cauchy-Schwarz Inequality

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$:

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$$

The Triangle Inequality

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$:

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Proof:

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot \vec{u} + 2\vec{u}\vec{v} + \vec{u} \cdot \vec{v} \\ &= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\ &\text{By the Cauchy-Schwarz Inequality} \\ &\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \\ &= (\|\vec{u}\| + \|\vec{v}\|)^2 \end{aligned}$$

Thus, $\|\vec{u}+\vec{v}\|^2 \leq (\|\vec{u}\|+\|\vec{v}\|)^2$. Taking the square root:

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\|+\|\vec{v}\|$$

Distance Formula

In $\mathbb{R}$, let $a, b \in \mathbb{R}$. Find the formula for $d(a, b)$:

$$d(a,b) = |a-b| = \sqrt{(a-b)^2}$$

In $\mathbb{R}^2$, let $\vec{v_1} = [a_1, b_1]$, and $v_2 = [a_2, b_2]$.

$$\begin{aligned} d(\vec{v_1}, \vec{v_2}) &= \|\vec{v_1} - \vec{v_2}\| \\ &= \sqrt{(a_1-a_2)^2 - (b_2-b_2)^2} \end{aligned}$$

In $\mathbb{R}^n$, let $\vec{v}, \vec{w}$ be vectors. Define the distance from $\vec{v}$ to $\vec{w}$ as follows:

$$\begin{aligned} d(\vec{v}, \vec{w}) &= \|\vec{v} - \vec{w}\| \\ &= \sqrt{\sum_{i=1}^{n} (v_1 - w_1)^2} \end{aligned}$$

Example

$$\begin{aligned} \vec{u} &= \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \\ d(\vec{u}, \vec{v}) &= \sqrt{(\sqrt{2}-0)^2 + (1-2)^2 + (-1+2)^2} \\ &= \sqrt{2 + (-1)^2 + 1^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Angle Between Vectors

If we have two vectors $\vec{v}, \vec{u}$ and the angle θ between them, we can use the law of cosines to find θ:

$$\begin{aligned}
\|\vec{u} - \vec{v}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta \\
&= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\
&= \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\
&= \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\
\|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta \\
-2\vec{u} \cdot \vec{v} &= -2\|\vec{u}\|\|\vec{v}\|\cos\theta \\
\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} &= \cos\theta
\end{aligned}$$

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$, the angle θ between $\vec{v}$ and $\vec{u}$ is:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Example

Find the angle between:

$$\begin{aligned}
\vec{u} &= \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
\vec{v} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
\vec{u} \cdot \vec{v} &= 2(1) + 1(1) + (-2)(1) \\
&= 2 + 1 - 2 = 1 \\
\|\vec{u}\| &= \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3 \\
\|\vec{v}\| &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \\
\cos\theta &= \vec{1}3\sqrt{3} \\
\theta &\approx 1.377rad \approx 78.9^\circ
\end{aligned}$$

Orthogonal Vectors

We say two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if there's a right angle between them.

$$\begin{aligned}
\cos(\frac{\pi}{2}) &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \\
0 &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \\
\therefore \vec{u}\cdot\vec{v} &= 0
\end{aligned}$$

Two vectors are orthogonal if their dot product is zero.

Example

$$\begin{aligned}
\vec{u} &= \langle 1, 1, -2\rangle \\
\vec{v} &= \langle 3, 1, 2\rangle \\
\vec{u}\cdot\vec{v} &= 1(3) + 1(1) + (-2)(2) \\
&= 3 + 1 - 4 = 0
\end{aligned}$$

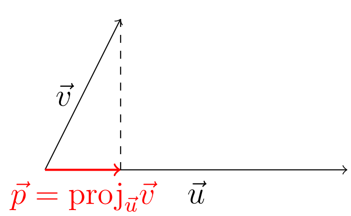
The Pythagorean Theorem

For vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, suppose $\vec{u}$ and $\vec{v}$ are orthogonal. Then:

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 \\
(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) &= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\
&= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 + 0 + \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2
\end{aligned}$$

Vector Projection

The projection of $\vec{v}$ onto $\vec{u}$ described by $\vec{p}$ is:



$$\text{proj}_{\vec{u}}\vec{v} = (\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}})\vec{u} = (\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2})\vec{u}$$

Practice Exercise 48

Given:

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \vec{v} = \begin{bmatrix} k+1 \\ k-1 \end{bmatrix}$$

Find all values of k such that $\vec{u}$ and $\vec{v}$ are orthogonal.

$$\begin{aligned} \vec{u}\cdot\vec{v} &= 0 \\ 2(k+1) + 3(k-1) &= 0 \\ 2k + 2 + 3k - 3 &= 0 \\ 5k - 1 &= 0 \\ k &= \frac{1}{5} \end{aligned}$$

Practice Exercise 49

Given:

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \vec{v} = \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$$

Find all values k such that $\vec{u}$ and $\vec{v}$ are orthogonal.

$$\begin{aligned} \vec{u}\cdot\vec{v} &= 0 \\ k^2 - k + 2(-3) &= 0 \\ k^2 - k - 6 &= 0 \\ (k-3)(k+2) &= 0 \\ k &= 3 \textit{ or} \\ k &= 2 \end{aligned}$$

Practice Exercise 50

Describe all vectors $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 0 \\ 3x + y &= 0 \\ y &= -3x \end{aligned}$$

y is a line going through the origin with a slope of -3.

Practice Exercise 52

Under what conditions are the following true for vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ *or* $\mathbb{R}^3$.

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\|\vec{u}\| + \|\vec{v}\|)^2 \\ (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) &= \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \\ \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} &= \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \\ \vec{u} \cdot \vec{v} &= \|\vec{u}\|\|\vec{v}\| \end{aligned}$$

Since:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

θ must be equal to zero since $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|$. Therefore, this can only be true for $\vec{u} = c\vec{v}, c > 0$.

Practice Exercise 55

Verify the stated property of distances:

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

$$\begin{aligned} d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \|(-1)(\vec{v} - \vec{u})\| \\ &= |-1|\|\vec{v} - \vec{u}\| \\ &= d(\vec{v}, \vec{u}) \end{aligned}$$

Practice Exercise 56

$$d(\vec{u},\vec{w}) \leq d(\vec{u},\vec{v}) + d(\vec{v},\vec{w})$$

$$\begin{aligned} d(\vec{u},\vec{w}) &= \|\vec{u}-\vec{w}\| \\ &= \|(\vec{u}-\vec{v}) + (\vec{v}-\vec{w})\| \\ &\leq \|\vec{u}-\vec{w}\| + \|\vec{v}+\vec{w}\| \\ &= d(\vec{u},\vec{v}) + d(\vec{v},\vec{w}) \end{aligned}$$

Practice Exercise 58

Prove:

$$\vec{u} \cdot c\vec{v} = c(\vec{u} \cdot \vec{v})$$

$$\begin{aligned} \vec{u} \cdot (c\vec{v}) &= (c\vec{v}) \cdot \vec{u} \\ &= c(\vec{v} \cdot \vec{u}) \\ &= c(\vec{u} \cdot \vec{v}) \end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech