Differential Equations: Homework 12

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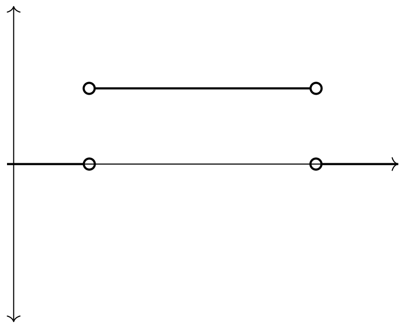
Section 7.6

Exercise 2

Sketch the graph of the given function and determine its Laplace transform.

$$u(t-1) - u(t-4)$$

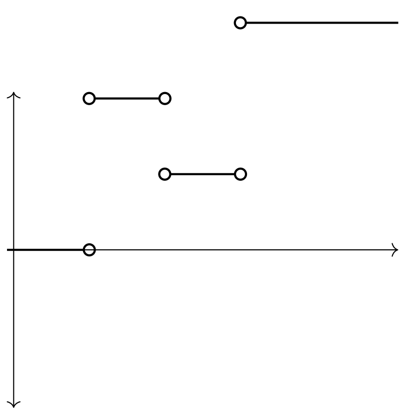


$$\begin{aligned}
\mathcal{L}\{u(t-1) - u(t-4)\} &= \mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-4)\} \\
&= \frac{e^{-s}}{s} - \frac{e^{-4s}}{s}
\end{aligned}$$

Exercise 5

Express the given function using window and step functions and compute its Laplace transform.

$$g(t) = \begin{cases} 0, & 0 < t < 1 \\ 2, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ 3, & 3 < t \end{cases}$$



$$\begin{aligned}
g(t) &= 0 + 2u(t-1) - u(t-2) + 2u(t-3) \\
\mathcal{L}\{g(t)\} &= 2\mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-2)\} + 2\mathcal{L}\{u(t-3)\} \\
&= \frac{2e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{2e^{-3s}}{s}
\end{aligned}$$

Exercise 6

Express the given function using window and step functions and compute its Laplace transform.

$$\begin{aligned}
g(t) &= \begin{cases} 0, & 0 < t < 2 \\ t+1, & 2 < t \end{cases} \\
&= (t+1)u(t-2) \\
\mathcal{L}\{g(t)\} &= \mathcal{L}\{(t+1)u(t-2)\} \\
&= \mathrm{e}^{-2s}\mathcal{L}\{(t+1)+2\} \\
&= \mathrm{e}^{-2s}\mathcal{L}\{t+3\} \\
&= \mathrm{e}^{-2s}(\mathcal{L}\{t\} + 3\mathcal{L}\{1\}) \\
&= \mathrm{e}^{-2s}(\frac{1}{s^2} + \frac{3}{s})
\end{aligned}$$

Exercise 11

Determine an inverse Laplace transform of the given function.

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}}{s-1}\} &= \mathcal{L}^{-1}\{\mathrm{e}^{-as}F(s)\} \\
&= f(t-a)u(t-a) \\
a &= 2 \\
F(s) &= \frac{1}{s-1} \\
f(t) &= \mathrm{e}^{t} \\
f(t-a)u(t-a) &= \mathrm{e}^{t-2}u(t-2)
\end{aligned}$$

Exercise 12

Determine an inverse Laplace transform of the given function.

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-3s}}{s^2}\} &= \mathcal{L}^{-1}\{\mathrm{e}^{-as}F(s)\} \\
&= f(t-a)u(t-a) \\
a &= 3 \\
F(s) &= \frac{1}{s^2} \\
f(t) &= t \\
f(t-a)u(t-a) &= (t-3)u(t-3)
\end{aligned}$$

Exercise 13

Determine an inverse Laplace transform of the given function.

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}-3\mathrm{e}^{-4s}}{s+2}\} &= \mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}}{s+2}\} - 3\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-4s}}{s+2}\} \\
a_1 &= 2 \\
F(s) &= \frac{1}{s+2} \\
f(t) &= \mathrm{e}^{-2t} \\
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}}{s+2}\} &= f(t-a_1)u(t-a_1) \\
&= \mathrm{e}^{-2(t-2)}u(t-2) \\
a_2 &= 4 \\
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-4s}}{s+2}\} &= f(t-a_2)u(t-a_2) \\
&= \mathrm{e}^{-2(t-4)}u(t-4) \\
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}}{s+2}\} - 3\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-4s}}{s+2}\} &= \mathrm{e}^{-2t+4}u(t-2) - 3\mathrm{e}^{-2t+8}u(t-4)
\end{aligned}$$

Exercise 14

Determine an inverse Laplace transform of the given function.

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-3s}}{s^2+9}\} &= \mathcal{L}^{-1}\{\mathrm{e}^{-as}F(s)\} \\
&= f(t-a)u(t-a) \\
a &= 3 \\
F(s) &= \frac{1}{s^2+9} \\
&= \frac{1}{3}\frac{3}{s^2+9} \\
f(t) &= \frac{1}{3}\sin(3t) \\
f(t-a)u(t-a) &= \frac{1}{3}\sin(3(t-3))u(t-3) \\
&= \frac{1}{3}\sin(3t-9)u(t-3)
\end{aligned}$$

Exercise 17

Determine an inverse Laplace transform of the given function.

$$
\begin{aligned}
\mathcal{L}^{-1}\{\frac{e^{-3s}(s-5)}{(s+1)(s+2)}\} &= \mathcal{L}^{-1}\{e^{-as}F(s)\} \\
&= f(t-a)u(t-a) \\
a &= 3 \\
F(s) &= \frac{s-5}{(s+1)(s+2)} \\
\frac{A}{s+1} + \frac{B}{s+2} &= \frac{s-5}{(s+1)(s+2)} \\
A(s+2) + B(s+1) &= s-5 \\
Let: s = -2 \quad B &= 7 \\
Let: s = -1 \quad A &= -6 \\
F(s) &= \frac{-6}{s+1} + \frac{7}{s+2} \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} \\
&= -6\mathcal{L}^{-1}\{\frac{1}{s+1}\} + 7\mathcal{L}^{-1}\{\frac{1}{s+2}\} \\
&= -6e^{-t} + 7e^{-2t} \\
f(t-a)u(t-a) &= (-6e^{-(t-3)} + 7e^{-2(t-3)})u(t-3) \\
&= (-6e^{-t+3} + 7e^{-2t+6})u(t-3)
\end{aligned}
$$

Exercise 21

Solve the given initial value problem using the method of Laplace transforms. Sketch the graph of the solution.

$$
\begin{aligned}
y'' + y &= u(t-3) \quad y(0) = 0 \quad y'(0) = 1 \\
\mathcal{L}\{y'' + y\} &= \mathcal{L}\{u(t-3)\} \\
s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{e^{-3s}}{s} \\
s^2Y(s) - 1 + Y(s) &= \frac{e^{-3s}}{s} \\
Y(s)(s^2+1) - 1 &= \frac{e^{-3s}}{s} \\
Y(s)(s^2+1) &= \frac{e^{-3s}}{s} + 1 \\
Y(s) &= \frac{e^{-3s}}{s(s^2+1)} + \frac{1}{s^2+1} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\{\frac{e^{-3s}}{s(s^2+1)}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-3s}}{s(s^2+1)}\} &= \mathcal{L}^{-1}\{\mathrm{e}^{-as}F(s)\} \\
a &= 3 \\
F(s) &= \frac{1}{s(s^2+1)} \\
&= \frac{A}{s} + \frac{Bs+C}{s^2+1} \\
A(s^2+1) + (Bs+C)s &= 1 \\
(A+B)s^2 + Cs + A &= 1 \\
A = 1 \quad A+B &= 0 \quad C = 0 \quad B = -1 \\
F(s) &= \frac{1}{s} - \frac{s}{s^2+1} \\
f(t) &= 1 - \cos(t) \\
f(t-a)u(t-a) &= (1-\cos(t-3))u(t-3) \\
y(t) &= (1-\cos(t-3))u(t-3) + \sin(t) \\
&= u(t-3) - \cos(t-3)u(t-3) + \sin(t)
\end{aligned}
$$

Exercise 29

Solve the given initial value problem using the method of Laplace transforms. Sketch the graph of the solution.

$$
\begin{aligned}
y'' + 4y &= g(t) \quad y(0) = 1 \quad y'(0) = 3 \\
g(t) &= \begin{cases} \sin(t), & 0 \le t \le 2\pi \\ 0, & 2\pi < t \end{cases} \\
&= \sin(t) - \sin(t)u(t-2\pi) \\
\mathcal{L}\{y'' + 4y\} &= \mathcal{L}\{g(t)\} \\
s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{1}{s^2+1} - \frac{\mathrm{e}^{-2\pi s}}{s^2+1} \\
s^2Y(s) - s - 3 + 4Y(s) &= \frac{1}{s^2+1} - \frac{\mathrm{e}^{-2\pi s}}{s^2+1} \\
Y(s)(s^2+4) &= \frac{1}{s^2+1} - \frac{\mathrm{e}^{-2\pi s}}{s^2+1} + s + 3 \\
Y(s) &= \frac{1}{(s^2+1)(s^2+4)} - \frac{\mathrm{e}^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s+3}{s^2+4} \\
&\text{partial fraction decomposition omitted for brevity} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
\mathcal{L}^{-1}\{\frac{1}{(s^2+1)(s^2+4)}\} &= \mathcal{L}^{-1}\{\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}\} \\
&= \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} - \frac{1}{6}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\} \\
&= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\} &= \mathcal{L}^{-1}\{e^{-as}F(s)\} \\
a &= 2\pi \\
F(s) &= \frac{1}{(s^2+1)(s^2+4)} \\
f(t) &= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t) \\
f(t-a)u(t-a) &= (\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2t-4\pi))u(t-2\pi) \\
&= \frac{u(t-2\pi)}{3}\sin(t) - \frac{u(t-2\pi)}{6}\sin(2t) \\
\mathcal{L}^{-1}\{\frac{s+3}{s^2+4}\} &= \mathcal{L}^{-1}\{\frac{s}{s^2+4}\} + \mathcal{L}^{-1}\{\frac{3}{s^2+4}\} \\
&= \mathcal{L}^{-1}\{\frac{s}{s^2+4}\} + \frac{3}{2}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\} \\
&= \cos(2t) + \frac{3}{2}\sin(2t) \\
y(t) &= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t) + \frac{u(t-2\pi)}{3}\sin(t) - \frac{u(t-2\pi)}{6}\sin(2t) + \cos(2t) + \frac{3}{2}\sin(2t) \\
&= \frac{1-u(t-2\pi)}{3}\sin(t) - \frac{8+u(t-2\pi)}{6}\sin(2t) + \cos(2t)
\end{aligned}
$$

Section 7.9

Exercise 5

Evaluate the given integral:

$$
\begin{aligned}
\int_0^\infty e^{-2t}\delta(t-1)\ dt &= \int_0^\infty f(t)\delta(t-1)\ dt \\
f(t) &= e^{-2t} \\
\int_0^\infty e^{-2t}\delta(t-1)\ dt &= f(a) \\
&= e^{-2}
\end{aligned}
$$

Exercise 7

Determine the Laplace transform of the given generalized function.

$$
\begin{aligned}
\mathcal{L}\{\delta(t-1) - \delta(t-3)\} &= \mathcal{L}\{\delta(t-1)\} - \mathcal{L}\{\delta(t-3)\} \\
&= e^{-s} - e^{-3s}
\end{aligned}
$$

Exercise 8

Determine the Laplace transform of the given generalized function.

$$
\begin{aligned}
\mathcal{L}\{3\delta(t-1)\} &= 3\mathcal{L}\{\delta(t-1)\} \\
&= 3e^{-s}
\end{aligned}
$$

Exercise 13

Solve the given symbolic initial value problem.

$$\begin{aligned}
w'' + w &= \delta(t-\pi) \quad w(0) = 0 \quad w'(0) = 0 \\
\mathcal{L}\{w'' + w\} &= \mathcal{L}\{\delta(t-\pi)\} \\
s^2W(s) - sw(0) - w'(0) + W(s) &= \mathrm{e}^{-\pi s} \\
W(s)(s^2+1) &= \mathrm{e}^{-\pi s} \\
W(s) &= \frac{\mathrm{e}^{-\pi s}}{s^2+1} \\
w(t) &= \mathcal{L}^{-1}\{\frac{\mathrm{e}^{-\pi s}}{s^2+1}\} \\
&= \mathcal{L}^{-1}\{\mathrm{e}^{-as}F(s)\} \\
a &= \pi \\
F(s) &= \frac{1}{s^2+1} \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = \sin(t) \\
w(t) &= f(t-a)u(t-a) = \sin(t-\pi)u(t-\pi)
\end{aligned}$$

Exercise 15

Solve the given symbolic initial value problem.

$$\begin{aligned}
y'' + 2y' - 3y &= \delta(t-1) - \delta(t-2) \quad y(0) = 2 \quad y'(0) = -2 \\
\mathcal{L}\{y'' + 2y' - 3y\} &= \mathcal{L}\{\delta(t-1) - \delta(t-2)\} \\
s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) - 3Y(s) &= \mathrm{e}^{-s} - \mathrm{e}^{-2s} \\
s^2Y(s) - 2s + 2 + 2sY(s) - 4 - 3Y(s) &= \mathrm{e}^{-s} - \mathrm{e}^{-2s} \\
Y(s)(s^2 + 2s - 3) - 2s - 2 &= \mathrm{e}^{-s} - \mathrm{e}^{-2s} \\
Y(s) &= \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s} + 2s + 2}{s^2 + 2s - 3}
\end{aligned}$$

$$y(t) = \mathcal{L}^{-1}\{\frac{\mathrm{e}^{-s}}{(s+3)(s-1)}\} - \mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}}{(s+3)(s-1)}\} + 2\mathcal{L}^{-1}\{\frac{s}{(s+3)(s-1)}\} + 2\mathcal{L}^{-1}\{\frac{1}{(s+3)(s-1)}\}$$

partial fraction decomposition omitted for brevity

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-s}}{(s+3)(s-1)}\} &= \frac{1}{4}\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-s}}{s-1}\} - \frac{1}{4}\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-s}}{s+3}\} \\
&= \frac{\mathrm{e}^{t-1}}{4}u(t-1) - \frac{\mathrm{e}^{3-t}}{4}u(t-1) \\
&= \frac{\mathrm{e}^{t-1} - \mathrm{e}^{3-t}}{4}u(t-1) \\
\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}}{(s+3)(s-1)}\} &= \frac{1}{4}\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}}{s-1}\} - \frac{1}{4}\mathcal{L}^{-1}\{\frac{\mathrm{e}^{-2s}}{s+3}\} \\
&= \frac{\mathrm{e}^{t-2}}{4}u(t-2) - \frac{\mathrm{e}^{6-3t}}{4}u(t-2) \\
&= \frac{\mathrm{e}^{t-2} - \mathrm{e}^{6-3t}}{4}u(t-2)
\end{aligned}$$

$$
\begin{aligned}
2\mathcal{L}^{-1}\{\frac{s}{(s+3)(s-1)}\} &= \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s-1}\} + \frac{3}{2}\mathcal{L}^{-1}\{\frac{1}{s+3}\} \\
&= \frac{e^t}{2} + \frac{3e^{-3t}}{2} \\
2\mathcal{L}^{-1}\{\frac{1}{(s+3)(s-1)}\} &= \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s-1}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s+3}\} \\
&= \frac{e^t}{2} - \frac{e^{-3t}}{2} \\
y(t) &= \frac{e^{t-1} - e^{3-t}}{4}u(t-1) - \frac{e^{t-2} - e^{6-3t}}{4}u(t-2) + \frac{e^t}{2} + \frac{3e^{-3t}}{2} + \frac{e^t}{2} - \frac{e^{-3t}}{2} \\
&= \frac{e^{t-1} - e^{3-t}}{4}u(t-1) - \frac{e^{t-2} - e^{6-3t}}{4}u(t-2) + e^t + e^{-3t}
\end{aligned}
$$

Exercise 16

Solve the given symbolic initial value problem.

$$
\begin{aligned}
y'' - 2y' - 3y &= 2\delta(t-1) - \delta(t-3) \quad y(0) = 2 \quad y'(0) = 2 \\
\mathcal{L}\{y'' - 2y' - 3y\} &= \mathcal{L}\{2\delta(t-1) - \delta(t-3)\} \\
s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) - 3Y(s) &= 2e^{-s} - e^{-3s} \\
s^2Y(s) - 2s - 2 - 2sY(s) + 4 - 3Y(s) &= 2e^{-s} - e^{-3s} \\
Y(s)(s^2 - 2s - 3) &= 2e^{-s} - e^{-3s} + 2s - 2 \\
Y(s) &= \frac{2e^{-s} - e^{-3s} + 2s - 2}{(s-3)(s+1)}
\end{aligned}
$$

$$
\begin{aligned}
Y(s) &= 2\frac{e^{-s}}{(s-3)(s+1)} - \frac{e^{-3s}}{(s-3)(s+1)} + 2\frac{s}{(s-3)(s+1)} - 2\frac{1}{(s-3)(s+1)} \\
&\text{partial fraction decomposition omitted for brevity} \\
&= \frac{1}{2}\frac{e^{-s}}{s-3} - \frac{1}{2}\frac{e^{-s}}{s+1} - \frac{1}{4}\frac{e^{-3s}}{s-3} + \frac{1}{4}\frac{e^{-3s}}{s+1} + \frac{3}{2}\frac{1}{s-3} + \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s-3} + \frac{1}{2}\frac{1}{s+1} \\
&= \frac{1}{2}\frac{e^{-s}}{s-3} - \frac{1}{2}\frac{e^{-s}}{s+1} - \frac{1}{2}\frac{e^{-3s}}{s-3} + \frac{1}{4}\frac{e^{-3s}}{s+1} + \frac{1}{s-3} + \frac{1}{s+1} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \frac{e^{3t-3}}{2}u(t-1) - \frac{e^{1-t}}{2}u(t-1) - \frac{e^{3t-9}}{4}u(t-3) - \frac{e^{3-t}}{4}u(t-3) + e^{3t} + e^{-t} \\
&= \frac{e^{3t-3} - e^{1-t}}{2}u(t-1) - \frac{e^{3t-9} - e^{3-t}}{4}u(t-3) + e^{3t} + e^{-t}
\end{aligned}
$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech