

# Differential Equations: Homework 12

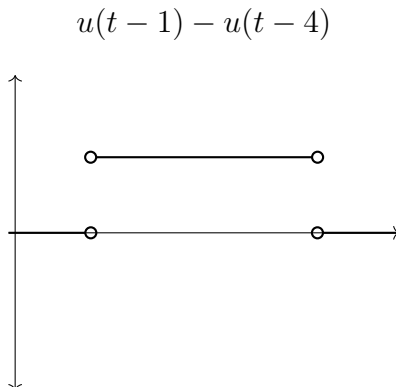
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## Section 7.6

### Exercise 2

Sketch the graph of the given function and determine its Laplace transform.

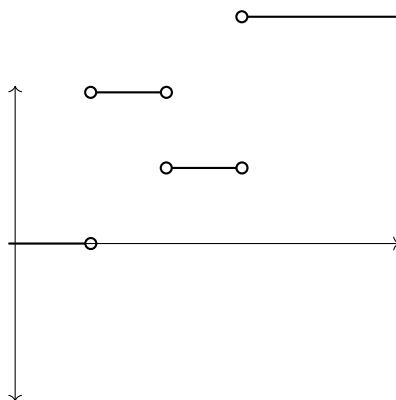


$$\begin{aligned} \mathcal{L}\{u(t-1) - u(t-4)\} &= \mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-4)\} \\ &= \frac{e^{-s}}{s} - \frac{e^{-4s}}{s} \end{aligned}$$

### Exercise 5

Express the given function using window and step functions and compute its Laplace transform.

$$g(t) = \begin{cases} 0, & 0 < t < 1 \\ 2, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ 3, & 3 < t \end{cases}$$



$$\begin{aligned} g(t) &= 0 + 2u(t-1) - u(t-2) + 2u(t-3) \\ \mathcal{L}\{g(t)\} &= 2\mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-2)\} + 2\mathcal{L}\{u(t-3)\} \\ &= \frac{2e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{2e^{-3s}}{s} \end{aligned}$$

**Exercise 6**

Express the given function using window and step functions and compute its Laplace transform.

$$\begin{aligned}
 g(t) &= \begin{cases} 0, & 0 < t < 2 \\ t + 1, & 2 < t \end{cases} \\
 &= (t + 1)u(t - 2) \\
 \mathcal{L}\{g(t)\} &= \mathcal{L}\{(t + 1)u(t - 2)\} \\
 &= e^{-2s}\mathcal{L}\{(t + 1) + 2\} \\
 &= e^{-2s}\mathcal{L}\{t + 3\} \\
 &= e^{-2s}(\mathcal{L}\{t\} + 3\mathcal{L}\{1\}) \\
 &= e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right)
 \end{aligned}$$

**Exercise 11**

Determine an inverse Laplace transform of the given function.

$$\begin{aligned}
 \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s - 1}\right\} &= \mathcal{L}^{-1}\{e^{-as}F(s)\} \\
 &= f(t - a)u(t - a) \\
 a &= 2 \\
 F(s) &= \frac{1}{s - 1} \\
 f(t) &= e^t \\
 f(t - a)u(t - a) &= e^{t-2}u(t - 2)
 \end{aligned}$$

**Exercise 12**

Determine an inverse Laplace transform of the given function.

$$\begin{aligned}
 \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} &= \mathcal{L}^{-1}\{e^{-as}F(s)\} \\
 &= f(t - a)u(t - a) \\
 a &= 3 \\
 F(s) &= \frac{1}{s^2} \\
 f(t) &= t \\
 f(t - a)u(t - a) &= (t - 3)u(t - 3)
 \end{aligned}$$

### Exercise 13

Determine an inverse Laplace transform of the given function.

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s} - 3e^{-4s}}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s+2}\right\}$$

$$a_1 = 2$$

$$F(s) = \frac{1}{s+2}$$

$$f(t) = e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\} = f(t - a_1)u(t - a_1)$$

$$= e^{-2(t-2)}u(t - 2)$$

$$a_2 = 4$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s+2}\right\} = f(t - a_2)u(t - a_2)$$

$$= e^{-2(t-4)}u(t - 4)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s+2}\right\} = e^{-2t+4}u(t - 2) - 3e^{-2t+8}u(t - 4)$$

### Exercise 14

Determine an inverse Laplace transform of the given function.

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2 + 9}\right\} = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

$$= f(t - a)u(t - a)$$

$$a = 3$$

$$F(s) = \frac{1}{s^2 + 9}$$

$$= \frac{1}{3} \frac{3}{s^2 + 9}$$

$$f(t) = \frac{1}{3} \sin(3t)$$

$$f(t - a)u(t - a) = \frac{1}{3} \sin(3(t - 3))u(t - 3)$$

$$= \frac{1}{3} \sin(3t - 9)u(t - 3)$$

**Exercise 17**

Determine an inverse Laplace transform of the given function.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{e^{-3s}(s-5)}{(s+1)(s+2)}\right\} &= \mathcal{L}^{-1}\{e^{-as}F(s)\} \\ &= f(t-a)u(t-a) \\ a &= 3 \\ F(s) &= \frac{s-5}{(s+1)(s+2)} \\ \frac{A}{s+1} + \frac{B}{s+2} &= \frac{s-5}{(s+1)(s+2)} \\ A(s+2) + B(s+1) &= s-5 \\ \text{Let : } s &= -2 \quad B = 7 \\ \text{Let : } s &= -1 \quad A = -6 \\ F(s) &= \frac{-6}{s+1} + \frac{7}{s+2} \\ f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= -6\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 7\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= -6e^{-t} + 7e^{-2t} \\ f(t-a)u(t-a) &= (-6e^{-(t-3)} + 7e^{-2(t-3)})u(t-3) \\ &= (-6e^{-t+3} + 7e^{-2t+6})u(t-3) \end{aligned}$$

**Exercise 21**

Solve the given initial value problem using the method of Laplace transforms. Sketch the graph of the solution.

$$\begin{aligned} y'' + y &= u(t-3) \quad y(0) = 0 \quad y'(0) = 1 \\ \mathcal{L}\{y'' + y\} &= \mathcal{L}\{u(t-3)\} \\ s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{e^{-3s}}{s} \\ s^2Y(s) - 1 + Y(s) &= \frac{e^{-3s}}{s} \\ Y(s)(s^2 + 1) - 1 &= \frac{e^{-3s}}{s} \\ Y(s)(s^2 + 1) &= \frac{e^{-3s}}{s} + 1 \\ Y(s) &= \frac{e^{-3s}}{s(s^2 + 1)} + \frac{1}{s^2 + 1} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2 + 1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} \end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

$$a = 3$$

$$\begin{aligned} F(s) &= \frac{1}{s(s^2+1)} \\ &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \end{aligned}$$

$$A(s^2+1) + (Bs+C)s = 1$$

$$(A+B)s^2 + Cs + A = 1$$

$$A = 1 \quad A+B = 0 \quad C = 0 \quad B = -1$$

$$F(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

$$f(t) = 1 - \cos(t)$$

$$f(t-a)u(t-a) = (1 - \cos(t-3))u(t-3)$$

$$y(t) = (1 - \cos(t-3))u(t-3) + \sin(t)$$

$$= u(t-3) - \cos(t-3)u(t-3) + \sin(t)$$

### Exercise 29

Solve the given initial value problem using the method of Laplace transforms. Sketch the graph of the solution.

$$y'' + 4y = g(t) \quad y(0) = 1 \quad y'(0) = 3$$

$$g(t) = \begin{cases} \sin(t), & 0 \leq t \leq 2\pi \\ 0, & 2\pi < t \end{cases}$$

$$= \sin(t) - \sin(t)u(t-2\pi)$$

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

$$s^2Y(s) - s - 3 + 4Y(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

$$Y(s)(s^2+4) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} + s + 3$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s+3}{s^2+4}$$

partial fraction decomposition omitted for brevity

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3} \frac{1}{s^2+1} - \frac{1}{3} \frac{1}{s^2+4}\right\}$$

$$= \frac{1}{3} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{6} \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{3} \sin(t) - \frac{1}{6} \sin(2t)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2 + 1)(s^2 + 4)}\right\} = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

$$a = 2\pi$$

$$F(s) = \frac{1}{(s^2 + 1)(s^2 + 4)}$$

$$f(t) = \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)$$

$$\begin{aligned} f(t-a)u(t-a) &= \left(\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2t-4\pi)\right)u(t-2\pi) \\ &= \frac{u(t-2\pi)}{3}\sin(t) - \frac{u(t-2\pi)}{6}\sin(2t) \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s+3}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \cos(2t) + \frac{3}{2}\sin(2t) \end{aligned}$$

$$\begin{aligned} y(t) &= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t) + \frac{u(t-2\pi)}{3}\sin(t) - \frac{u(t-2\pi)}{6}\sin(2t) + \cos(2t) + \frac{3}{2}\sin(2t) \\ &= \frac{1-u(t-2\pi)}{3}\sin(t) - \frac{8+u(t-2\pi)}{6}\sin(2t) + \cos(2t) \end{aligned}$$

## Section 7.9

### Exercise 5

Evaluate the given integral:

$$\begin{aligned} \int_0^{\infty} e^{-2t}\delta(t-1) dt &= \int_0^{\infty} f(t)\delta(t-1) dt \\ f(t) &= e^{-2t} \\ \int_0^{\infty} e^{-2t}\delta(t-1) dt &= f(a) \\ &= e^{-2} \end{aligned}$$

### Exercise 7

Determine the Laplace transform of the given generalized function.

$$\begin{aligned} \mathcal{L}\{\delta(t-1) - \delta(t-3)\} &= \mathcal{L}\{\delta(t-1)\} - \mathcal{L}\{\delta(t-3)\} \\ &= e^{-s} - e^{-3s} \end{aligned}$$

### Exercise 8

Determine the Laplace transform of the given generalized function.

$$\begin{aligned} \mathcal{L}\{3\delta(t-1)\} &= 3\mathcal{L}\{\delta(t-1)\} \\ &= 3e^{-s} \end{aligned}$$

### Exercise 13

Solve the given symbolic initial value problem.

$$\begin{aligned}
w'' + w &= \delta(t - \pi) & w(0) = 0 & \quad w'(0) = 0 \\
\mathcal{L}\{w'' + w\} &= \mathcal{L}\{\delta(t - \pi)\} \\
s^2W(s) - sw(0) - w'(0) + W(s) &= e^{-\pi s} \\
W(s)(s^2 + 1) &= e^{-\pi s} \\
W(s) &= \frac{e^{-\pi s}}{s^2 + 1} \\
w(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2 + 1}\right\} \\
&= \mathcal{L}^{-1}\{e^{-as}F(s)\} \\
a &= \pi \\
F(s) &= \frac{1}{s^2 + 1} \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = \sin(t) \\
w(t) &= f(t - a)u(t - a) = \sin(t - \pi)u(t - \pi)
\end{aligned}$$

### Exercise 15

Solve the given symbolic initial value problem.

$$\begin{aligned}
y'' + 2y' - 3y &= \delta(t - 1) - \delta(t - 2) & y(0) = 2 & \quad y'(0) = -2 \\
\mathcal{L}\{y'' + 2y' - 3y\} &= \mathcal{L}\{\delta(t - 1) - \delta(t - 2)\} \\
s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) - 3Y(s) &= e^{-s} - e^{-2s} \\
s^2Y(s) - 2s + 2 + 2sY(s) - 4 - 3Y(s) &= e^{-s} - e^{-2s} \\
Y(s)(s^2 + 2s - 3) - 2s - 2 &= e^{-s} - e^{-2s} \\
Y(s) &= \frac{e^{-s} - e^{-2s} + 2s + 2}{s^2 + 2s - 3} \\
y(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+3)(s-1)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+3)(s-1)}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s-1)}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}
\end{aligned}$$

partial fraction decomposition omitted for brevity

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+3)(s-1)}\right\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s-1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+3}\right\} \\
&= \frac{e^{t-1}}{4}u(t-1) - \frac{e^{3-t}}{4}u(t-1) \\
&= \frac{e^{t-1} - e^{3-t}}{4}u(t-1) \\
\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+3)(s-1)}\right\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+3}\right\} \\
&= \frac{e^{t-2}}{4}u(t-2) - \frac{e^{6-3t}}{4}u(t-2) \\
&= \frac{e^{t-2} - e^{6-3t}}{4}u(t-2)
\end{aligned}$$



$$\begin{aligned}
2\mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s-1)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\
&= \frac{e^t}{2} + \frac{3e^{-3t}}{2} \\
2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\
&= \frac{e^t}{2} - \frac{e^{-3t}}{2} \\
y(t) &= \frac{e^{t-1} - e^{3-t}}{4}u(t-1) - \frac{e^{t-2} - e^{6-3t}}{4}u(t-2) + \frac{e^t}{2} + \frac{3e^{-3t}}{2} + \frac{e^t}{2} - \frac{e^{-3t}}{2} \\
&= \frac{e^{t-1} - e^{3-t}}{4}u(t-1) - \frac{e^{t-2} - e^{6-3t}}{4}u(t-2) + e^t + e^{-3t}
\end{aligned}$$

### Exercise 16

Solve the given symbolic initial value problem.

$$\begin{aligned}
y'' - 2y' - 3y &= 2\delta(t-1) - \delta(t-3) \quad y(0) = 2 \quad y'(0) = 2 \\
\mathcal{L}\{y'' - 2y' - 3y\} &= \mathcal{L}\{2\delta(t-1) - \delta(t-3)\} \\
s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) - 3Y(s) &= 2e^{-s} - e^{-3s} \\
s^2Y(s) - 2s - 2 - 2sY(s) + 4 - 3Y(s) &= 2e^{-s} - e^{-3s} \\
Y(s)(s^2 - 2s - 3) &= 2e^{-s} - e^{-3s} + 2s - 2 \\
Y(s) &= \frac{2e^{-s} - e^{-3s} + 2s - 2}{(s-3)(s+1)}
\end{aligned}$$

$$Y(s) = 2\frac{e^{-s}}{(s-3)(s+1)} - \frac{e^{-3s}}{(s-3)(s+1)} + 2\frac{s}{(s-3)(s+1)} - 2\frac{1}{(s-3)(s+1)}$$

partial fraction decomposition omitted for brevity

$$\begin{aligned}
&= \frac{1}{2}\frac{e^{-s}}{s-3} - \frac{1}{2}\frac{e^{-s}}{s+1} - \frac{1}{4}\frac{e^{-3s}}{s-3} + \frac{1}{4}\frac{e^{-3s}}{s+1} + \frac{3}{2}\frac{1}{s-3} + \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s-3} + \frac{1}{2}\frac{1}{s+1} \\
&= \frac{1}{2}\frac{e^{-s}}{s-3} - \frac{1}{2}\frac{e^{-s}}{s+1} - \frac{1}{2}\frac{e^{-3s}}{s-3} + \frac{1}{4}\frac{e^{-3s}}{s+1} + \frac{1}{s-3} + \frac{1}{s+1} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \frac{e^{3t-3}}{2}u(t-1) - \frac{e^{1-t}}{2}u(t-1) - \frac{e^{3t-9}}{4}u(t-3) - \frac{e^{3-t}}{4}u(t-3) + e^{3t} + e^{-t} \\
&= \frac{e^{3t-3} - e^{1-t}}{2}u(t-1) - \frac{e^{3t-9} - e^{3-t}}{4}u(t-3) + e^{3t} + e^{-t}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)