

Differential Equations: Homework 11

Alvin Lin

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Section 7.5

Exercise 2

Solve the given initial value problem using the method of Laplace transforms.

$$y'' - y' - 2y = 0 \quad y(0) = -2 \quad y'(0) = 5$$

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

$$\left[s^2 Y(s) - sy(0) - y'(0) \right] - \left[sY(s) - y(0) \right] - 2(Y(s)) = 0$$

$$s^2 Y(s) + 2s - 5 - sY(s) - 2 - 2Y(s) = 0$$

$$s^2 Y(s) - sY(s) - 2Y(s) = 7 - 2s$$

$$Y(s)(s^2 - s - 2) = 7 - 2s$$

$$Y(s) = \frac{7 - 2s}{s^2 - s - 2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\frac{A}{s-2} + \frac{B}{s+1} = \frac{7-2s}{(s-2)(s+1)}$$

$$A(s+1) + B(s-2) = 7-2s$$

$$\text{Let : } s = -1 \quad B = -3$$

$$\text{Let : } s = 2 \quad A = 1$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{3}{s+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= e^{2t} - 3e^{-t} \end{aligned}$$

Exercise 3

Solve the given initial value problem using the method of Laplace transforms.

$$\begin{aligned}y'' + 6y' + 9y &= 0 \quad y(0) = -1 \quad y'(0) = 6 \\ \mathcal{L}\{y'' + 6y' + 9y\} &= \mathcal{L}\{0\} \\ \mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} &= 0 \\ \left[s^2Y(s) - sy(0) - y'(0) \right] + 6 \left[sY(s) - y(0) \right] + 9(Y(s)) &= 0 \\ s^2Y(s) + s - 6 + 6sY(s) + 6 + 9Y(s) &= 0 \\ s^2Y(s) + 6sY(s) + 9Y(s) &= -s \\ Y(s)(s^2 + 6s + 9) &= -s \\ Y(s) &= \frac{-s}{s^2 + 6s + 9} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ \frac{A}{(s+3)^2} + \frac{B}{s+3} &= \frac{-s}{(s+3)^2} \\ A + B(s+3) &= -s \\ \text{Let : } s = -3 \quad A &= 3 \\ \text{Let : } s = 1 \quad B &= -1 \\ y(t) &= \mathcal{L}^{-1}\left\{\frac{3}{(s+3)^2} - \frac{1}{s+3}\right\} \\ &= 3\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\ &= 3te^{-3t} - e^{-3t}\end{aligned}$$

Exercise 4

Solve the given initial value problem using the method of Laplace transforms.

$$\begin{aligned}y'' + 6y' + 5y &= 12e^t \quad y(0) = -1 \quad y'(0) = 7 \\ \mathcal{L}\{y'' + 6y' + 5y\} &= \mathcal{L}\{12e^t\} \\ \mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} &= 12\mathcal{L}\{e^t\} \\ \left[s^2Y(s) - sy(0) - y'(0) \right] + 6 \left[sY(s) - y(0) \right] + 5Y(s) &= 12\frac{1}{s-1} \\ s^2Y(s) + s - 7 + 6sY(s) + 6 + 5Y(s) &= \frac{12}{s-1} \\ s^2Y(s) + 6sY(s) + 5Y(s) &= \frac{12}{s-1} - s + 1 \\ Y(s)(s^2 + 6s + 5) &= \frac{12}{s-1} - s + 1 \\ Y(s) &= \frac{12}{(s-1)(s+5)(s+1)} - \frac{s-1}{(s+5)(s+1)} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ \frac{A}{s-1} + \frac{B}{s+5} + \frac{C}{s+1} &= \frac{12}{(s-1)(s+5)(s+1)}\end{aligned}$$

$$A(s+5)(s+1) + B(s-1)(s+1) + C(s-1)(s+5) = 12$$

$$\text{Let : } s = -5 \quad B = \frac{1}{2}$$

$$\text{Let : } s = -1 \quad C = -\frac{3}{2}$$

$$\text{Let : } s = 1 \quad A = 1$$

$$\frac{D}{s+5} + \frac{E}{s+1} = \frac{s-1}{(s+5)(s+1)}$$

$$D(s+1) + E(s+5) = s-1$$

$$\text{Let : } s = -5 \quad D = \frac{3}{2}$$

$$\text{Let : } s = -1 \quad E = -\frac{1}{2}$$

$$Y(s) = \frac{1}{s-1} + \frac{1}{2} \frac{1}{s+5} - \frac{3}{2} \frac{1}{s+1} - \frac{3}{2} \frac{1}{s+5} + \frac{1}{2} \frac{1}{s+1}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+5} - \frac{1}{s+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= e^t - e^{-5t} - e^{-t} \end{aligned}$$

Exercise 7

Solve the given initial value problem using the method of Laplace transforms.

$$y'' - 7y' + 10y = 9 \cos(t) + 7 \sin(t) \quad y(0) = 5 \quad y'(0) = -4$$

$$\mathcal{L}\{y'' - 7y' + 10y\} = \mathcal{L}\{9 \cos(t) + 7 \sin(t)\}$$

$$\mathcal{L}\{y''\} - 7\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 9\mathcal{L}\{\cos(t)\} + 7\mathcal{L}\{\sin(t)\}$$

$$\left[s^2 Y(s) - sy(0) - y'(0) \right] - 7 \left[sY(s) - y(0) \right] + 10Y(s) = 9 \frac{s}{s^2+1} + 7 \frac{1}{s^2+1}$$

$$s^2 Y(s) - 5s + 4 - 7sY(s) + 35 + 10Y(s) = \frac{9s+7}{s^2+1}$$

$$s^2 Y(s) - 7sY(s) + 10Y(s) = \frac{9s+7}{s^2+1} + 5s - 39$$

$$Y(s) = \frac{9s+7}{(s^2+1)(s-2)(s-5)} + \frac{5s-39}{(s-2)(s-5)}$$

$$= \frac{5s^3 - 39s^2 + 14s - 32}{(s^2+1)(s-2)(s-5)}$$

$$\frac{As+B}{s^2+1} + \frac{C}{s-2} + \frac{D}{s-5} = \frac{5s^3 - 39s^2 + 14s - 32}{(s^2+1)(s-2)(s-5)}$$

$$(As+B)(s-2)(s-5) + C(s^2+1)(s-5) + D(s^2+1)(s-2) = 5s^3 - 39s^2 + 14s - 32$$

$$\text{Let : } s = 2 \quad C = 8$$

$$\text{Let : } s = 5 \quad D = -4$$

$$\text{Let : } s = 0 \quad B = 0$$

$$\text{Let : } s = 1 \quad A = 1$$

$$Y(s) = \frac{s}{s^2 + 1} + \frac{8}{s - 2} - \frac{4}{s - 5}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} + 8\mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s - 5}\right\} \\ &= \cos(t) + 8e^{2t} - 4e^{5t} \end{aligned}$$

Exercise 10

Solve the given initial value problem using the method of Laplace transforms.

$$y'' - 4y = 4t - 8e^{-2t} \quad y(0) = 0 \quad y'(0) = 5$$

$$\mathcal{L}\{y'' - 4y\} = \mathcal{L}\{4t - 8e^{-t}\}$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y\} = 4\mathcal{L}\{t\} - 8\mathcal{L}\{e^{-t}\}$$

$$\left[s^2 Y(s) - sy(0) - y'(0) \right] - 4Y(s) = 4\frac{1}{s^2} - 8\frac{1}{s + 1}$$

$$s^2 Y(s) - 5 - 4Y(s) = \frac{4}{s^2} - \frac{8}{s + 1}$$

$$Y(s)(s^2 - 4) = \frac{4}{s^2} - \frac{8}{s + 2} + 5$$

$$Y(s) = \frac{4}{s^2(s^2 - 4)} - \frac{8}{(s + 2)(s^2 - 4)} + \frac{5}{s^2 - 4}$$

$$= \frac{5s^3 + 2s^2 + 4s + 8}{s^2(s^2 - 4)(s + 2)}$$

$$= \frac{5s^3 + 2s^2 + 4s + 8}{s^2(s + 2)(s - 2)(s + 2)}$$

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 2} + \frac{D}{(s + 2)^2} + \frac{E}{s - 2} = \frac{5s^3 + 2s^2 + 4s + 8}{s^2(s + 2)(s - 2)(s + 2)}$$

$$As(s + 2)^2(s - 2) + B(s + 2)^2(s - 2) + Cs^2(s + 2)(s - 2) + Ds^2(s - 2) + Es^2(s + 2)^2 = 5s^3 + 2s^2 + 4s + 8$$

$$\text{Let : } s = -2 \quad D = 2$$

$$\text{Let : } s = 2 \quad E = 1$$

$$\text{Let : } s = 0 \quad B = -1$$

$$A = 0 \quad \text{via system of equations}$$

$$C = -1$$

$$Y(s) = -\frac{1}{s^2} - \frac{1}{s + 2} + \frac{2}{(s + 2)^2} + \frac{1}{s - 2}$$

$$\begin{aligned} y(t) &= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\} \\ &= -t - e^{-2t} + 2te^{-2t} + e^{2t} \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech