

# Differential Equations: Homework 9

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## Section 4.10

### Exercise 4

Determine the equation of motion for an undamped system at resonance governed by:

$$\frac{d^2y}{dt^2} + y = 5 \cos(t) \quad y(0) = 0 \quad y'(0) = 1$$

Sketch the solution.

$$y'' + y = 5 \cos(t)$$

$$r^2 + 1 = 0$$

$$r = 0 \pm i \quad \alpha = 0 \quad \beta = 1$$

$$y_h = c_1 \cos(t) + c_2 \sin(t)$$

$$y_p = t \left[ A \cos(t) + B \sin(t) \right] = At \cos(t) + Bt \sin(t)$$

$$y'_p = A \cos(t) - At \sin(t) + B \sin(t) + Bt \cos(t)$$

$$y''_p = -A \sin(t) - \left[ A \sin(t) + At \cos(t) \right] + B \cos(t) + \left[ B \cos(t) - Bt \sin(t) \right]$$

$$y'' + y = 5 \cos(t)$$

$$= -A \sin(t) - A \sin(t) - At \cos(t) + B \cos(t) + B \cos(t) -$$

$$Bt \sin(t) + At \cos(t) + Bt \sin(t)$$

$$= -2A \sin(t) + 2B \cos(t)$$

$$-2A = 0 \quad 2B = 5 \quad A = 0 \quad B = \frac{5}{2}$$

$$y_p = \frac{5}{2}t \sin(t)$$

$$y = c_1 \cos(t) + c_2 \sin(t) + \frac{5}{2}t \sin(t)$$

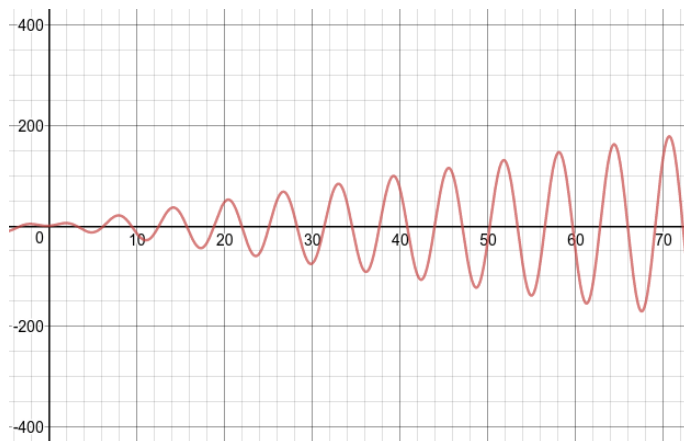
$$y' = -c_1 \sin(t) + c_2 \cos(t) + c_2 \cos(t) + \frac{5}{2} \sin(t) + \frac{5}{2}t \cos(t)$$

$$y(0) = 0 = c_1$$

$$y'(0) = 1 = c_2$$

$$y = \sin(t) + \frac{5}{2}t \sin(t)$$

$$= \left(1 + \frac{5}{2}t\right) \sin(t)$$



### Exercise 15

An 8-kg mass is attached to a string hanging from the ceiling and allowed to come to rest. Assume that the spring constant is  $40\frac{N}{m}$  and the damping constant is  $3\frac{N\cdot sec}{m}$ . At time  $t = 0$ , an external force of  $2\sin(2t)\cos(2t)$  N is applied to the system. Determine the amplitude and frequency of the steady-state solution.

$$8y'' + 3y' + 40y = 2\sin(2t)\cos(2t)$$

$$8r^2 + 3r + 40 = 0$$

$$r = \frac{-3 \pm \sqrt{9 - 4(8)(40)}}{2(8)}$$

$$= -\frac{3}{16} \pm \frac{\sqrt{1271}i}{16} \quad \alpha = -\frac{3}{16} \quad \beta = \frac{\sqrt{1271}}{16}$$

$$y_h = e^{-\frac{3}{16}t} \left( c_1 \cos\left(\frac{\sqrt{1271}}{16}t\right) + c_2 \sin\left(\frac{\sqrt{1271}}{16}t\right) \right)$$

$$8y'' + 3y' + 40y = 2\sin(2t)\cos(2t) = \sin(4t)$$

$$y_p = A \cos(4t) + B \sin(4t)$$

$$y_p' = -4A \sin(4t) + 4B \cos(4t)$$

$$y_p'' = -16A \cos(4t) - 16B \sin(4t)$$

$$8(-16A \cos(4t) - 16B \sin(4t)) + 3(-4A \sin(4t) + 4B \cos(4t)) + 40(A \cos(4t) + B \sin(4t)) = \sin(4t)$$

$$(-88A + 12B) \cos(4t) + (-12A - 88B) \sin(4t) = \sin(4t)$$

$$-12A - 88B = 1 \quad -88A + 12B = 0$$

$$A = \frac{3}{1972} \quad B = \frac{11}{986}$$

$$y_p = \frac{3}{1972} \cos(4t) + \frac{11}{986} \sin(4t)$$

$$\text{amplitude} = \sqrt{\left(\frac{3}{1972}\right)^2 + \left(\frac{11}{986}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{B}{A}\right) \approx 82.23$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$f = \frac{\omega}{2\pi} = \frac{2}{\pi}$$

## Section 7.2

### Exercise 3

Use Definition 1 to determine the Laplace transform of the given function.

$$\begin{aligned}\mathcal{L}\{e^{6t}\} &= \int_0^{\infty} e^{-st} e^{6t} dt \\ &= \lim_{N \rightarrow \infty} \int_0^N e^{6t-st} dt \\ &= \lim_{N \rightarrow \infty} \int_0^N e^{(6-s)t} dt \\ &= \lim_{N \rightarrow \infty} \left[ \frac{1}{6-s} e^{(6-s)t} \right]_0^N \\ &= 0 - \frac{1}{6-s}, \quad s > 6\end{aligned}$$

### Exercise 11

Use Definition 1 to determine the Laplace transform of the given function.

$$\begin{aligned}f(t) &= \begin{cases} \sin(t), & 0 < t < \pi \\ 0, & \pi < t \end{cases} \\ \mathcal{L}\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\pi} e^{-st} \sin(t) dt + \int_{\pi}^{\infty} e^{-st} 0 dt \\ &= \int_0^{\pi} e^{-st} \sin(t) dt \\ &= \left[ -\frac{e^{-st}(s \sin(t) + \cos(t))}{s^2 + 1} \right]_0^{\pi} \\ &= \frac{-e^{-s\pi}(-1)}{s^2 + 1} - \frac{-e^0(1)}{s^2 + 1} \\ &= \frac{e^{-s\pi} + 1}{s^2 + 1}\end{aligned}$$

### Exercise 13

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{6e^{-3t} - t^2 + 2t - 8\} &= 6\mathcal{L}\{e^{-3t}\} - \mathcal{L}\{t^2\} + 2\mathcal{L}\{t\} - 8\mathcal{L}\{1\} \\ &= 6\left(\frac{1}{s+3}\right) - \frac{2!}{s^3} + 2\left(\frac{1}{s^2}\right) - 8\left(\frac{1}{s}\right) \\ &= \frac{6}{s+3} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{8}{s}, \quad s > 0\end{aligned}$$

### Exercise 14

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{5 - e^{2t} + 6t^2\} &= 5\mathcal{L}\{1\} - \mathcal{L}\{e^{2t}\} + 6\mathcal{L}\{t^2\} \\ &= 5\left(\frac{1}{s}\right) - \frac{1}{s-2} + 6\left(\frac{2!}{s^3}\right) \\ &= \frac{5}{s} - \frac{1}{s-2} + \frac{12}{s^3}, \quad s > 2\end{aligned}$$

### Exercise 15

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{t^3 - te^t + e^{4t} \cos(t)\} &= \mathcal{L}\{t^3\} - \mathcal{L}\{te^t\} + \mathcal{L}\{e^{4t} \cos(t)\} \\ &= \frac{3!}{s^4} - \frac{1}{(s-1)^2} + \frac{s-4}{(s-4)^2 + 1^2} \\ &= \frac{6}{s^4} - \frac{1}{(s-2)^2} + \frac{s-4}{(s-4)^2 + 1}, \quad s > 4\end{aligned}$$

### Exercise 16

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{t^2 - 3t - 2e^{-t} \sin(3t)\} &= \mathcal{L}\{t^2\} - 3\mathcal{L}\{t\} - 2\mathcal{L}\{e^{-t} \sin(3t)\} \\ &= \frac{2}{s^3} - 3\left(\frac{1}{s^2}\right) - 2\left(\frac{3}{(s+1)^2 + 3^2}\right) \\ &= \frac{2}{s^3} - \frac{3}{s^2} - \frac{6}{(s+1)^2 + 9}, \quad s > 0\end{aligned}$$

### Exercise 17

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{e^{3t} \sin(6t) - t^3 + e^t\} &= \mathcal{L}\{e^{3t} \sin(6t)\} - \mathcal{L}\{t^3\} + \mathcal{L}\{e^t\} \\ &= \frac{6}{(s-3)^2 + 6^2} - \frac{3!}{s^4} + \frac{1}{s-1} \\ &= \frac{6}{(s-3)^2 + 36} - \frac{6}{s^4} + \frac{1}{s-1}, \quad s > 3\end{aligned}$$

**Exercise 18**

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{t^4 - t^2 - t + \sin(\sqrt{2}t)\} &= \mathcal{L}\{t^4\} - \mathcal{L}\{t^2\} - \mathcal{L}\{t\} + \mathcal{L}\{\sin(\sqrt{2}t)\} \\ &= \frac{4!}{s^5} - \frac{2!}{s^3} - \frac{1!}{s^2} + \frac{\sqrt{2}}{s^2 + (\sqrt{2})^2} \\ &= \frac{24}{s^5} - \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2 + 2}, \quad s > 0\end{aligned}$$

**Exercise 19**

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{t^4 e^{5t} - e^t \cos(\sqrt{7}t)\} &= \mathcal{L}\{t^4 e^{5t}\} - \mathcal{L}\{e^t \cos(\sqrt{7}t)\} \\ &= \frac{4!}{(s-5)^5} - \frac{s-1}{(s-1)^2 + (\sqrt{7})^2} \\ &= \frac{24}{(s-5)^5} - \frac{s-1}{(s-1)^2 + 7}, \quad s < 5\end{aligned}$$

**Exercise 20**

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

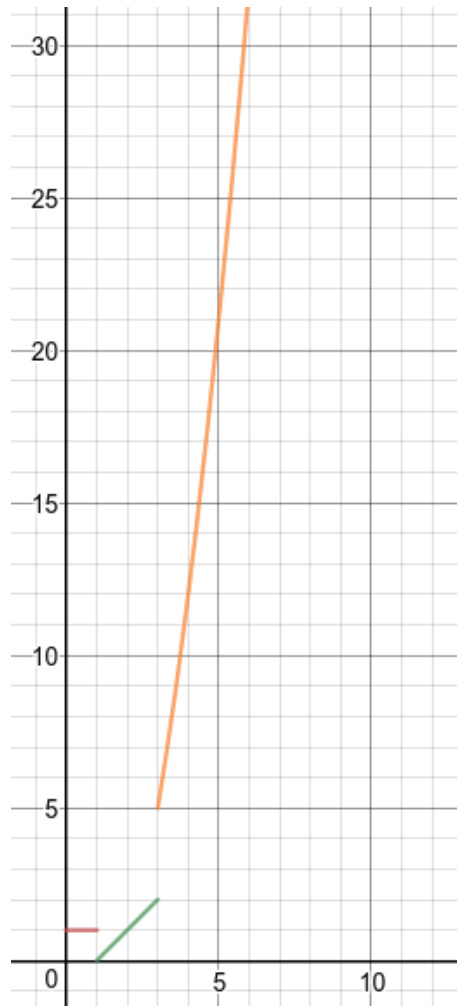
$$\begin{aligned}\mathcal{L}\{e^{-2t} \cos(\sqrt{3}t) - t^2 e^{-2t}\} &= \mathcal{L}\{e^{-2t} \cos(\sqrt{3}t)\} - \mathcal{L}\{t^2 e^{-2t}\} \\ &= \frac{s+2}{(s+2)^2 + (\sqrt{3})^2} - \frac{2!}{(s+2)^3} \\ &= \frac{s+2}{(s+2)^2 + 3} - \frac{2}{(s+2)^3}, \quad s > -2\end{aligned}$$

**Exercise 23**

Determine whether  $f(t)$  is continuous, piecewise continuous, or neither on  $[0, 10]$  and sketch the graph of  $f(t)$ .

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t - 1, & 1 < t < 3 \\ t^2 - 4, & 3 < t \leq 10 \end{cases}$$

This function is piecewise continuous.

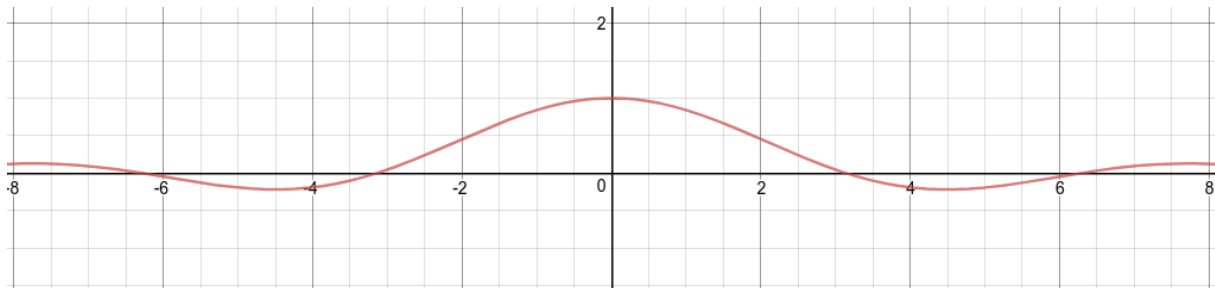


**Exercise 28**

Determine whether  $f(t)$  is continuous, piecewise continuous, or neither on  $[0, 10]$  and sketch the graph of  $f(t)$ .

$$f(t) = \begin{cases} \frac{\sin(t)}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

$f(t)$  is continuous and piecewise continuous.



**Exercise 30**

For the transforms  $F(s)$  in Table 7.1, what can be said about  $\lim_{s \rightarrow \infty} F(s)$ ?

$s$  is mostly in the denominator and increases in such a way that as  $s \rightarrow \infty$ ,  $F(s)$  goes to 0.

If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)