Differential Equations: Homework 9

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Section 4.10

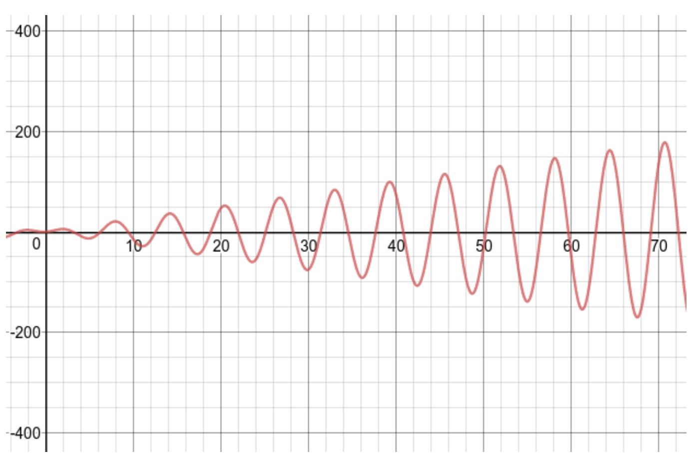
Exercise 4

Determine the equation of motion for an undamped system at resonance governed by:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y = 5\cos(t) \quad y(0) = 0 \quad y'(0) = 1$$

Sketch the solution.

$$\begin{aligned}
y'' + y &= 5\cos(t) \\
r^2 + 1 &= 0 \\
r &= 0 \pm i \quad \alpha = 0 \quad \beta = 1 \\
y_h &= c_1\cos(t) + c_2\sin(t) \\
y_p &= t\Big[A\cos(t) + B\sin(t)\Big] = At\cos(t) + Bt\cos(t) \\
y_p' &= A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t) \\
y_p'' &= -A\sin(t) - \Big[A\sin(t) + At\cos(t)\Big] + B\cos(t) + \Big[B\cos(t) - Bt\sin(t)\Big] \\
y'' + y &= 5\cos(t) \\
&= -A\sin(t) - A\sin(t) - At\cos(t) + B\cos(t) + B\cos(t) - \\
&\quad Bt\sin(t) + At\cos(t) + Bt\sin(t) \\
&= -2A\sin(t) + 2B\cos(t) \\
-2A &= 0 \quad 2B = 5 \quad A = 0 \quad B = \frac{5}{2} \\
y_p &= \frac{5}{2}t\sin(t) \\
y &= c_1\cos(t) + c_2\sin(t) + \frac{5}{2}t\sin(t) \\
y' &= -c_1\sin(t) + c_2\cos(t) + c_2\cos(t) + \frac{5}{2}\sin(t) + \frac{5}{2}t\cos(t) \\
y(0) &= 0 = c_1 \\
y'(0) &= 1 = c_2 \\
y &= \sin(t) + \frac{5}{2}t\sin(t) \\
&= (1 + \frac{5}{2}t)\sin(t)
\end{aligned}$$



Exercise 15

An 8-kg mass is attached to a string hanging from the ceiling and allowed to come to rest. Assume that the spring constant is $40\frac{N}{m}$ and the damping constant is $3\frac{N\cdot sec}{m}$. At time $t = 0$, an external force of $2\sin(2t)\cos(2t)$ N is applied to the system. Determine the amplitude and frequency of the steady-state solution.

$$\begin{aligned}
8y'' + 3y' + 40y &= 2\sin(2t)\cos(2t) \\
8r^2 + 3r + 40 &= 0 \\
r &= \frac{-3 \pm \sqrt{9 - 4(8)(40)}}{2(8)} \\
&= -\frac{3}{16} \pm \frac{\sqrt{1271}i}{16} \quad \alpha = -\frac{3}{16} \quad \beta = \frac{\sqrt{1271}}{16} \\
y_h &= \mathrm{e}^{-\frac{3}{16}t}\left(c_1\cos(\frac{\sqrt{1271}}{16}t) + c_2\sin(\frac{\sqrt{1271}}{16}t)\right) \\
8y'' + 3y' + 40y &= 2\sin(2t)\cos(2t) = \sin(4t) \\
y_p &= A\cos(4t) + B\sin(4t) \\
y_p' &= -4A\sin(4t) + 4B\cos(4t) \\
y_p'' &= -16A\cos(4t) - 16B\sin(4t)
\end{aligned}$$

$$\begin{aligned}
8(-16A\cos(4t) - 16B\sin(4t)) + 3(-4A\sin(4t) + 4B\cos(4t)) + 40(A\cos(4t) + B\sin(4t)) &= \sin(4t) \\
(-88A + 12B)\cos(4t) + (-12A - 88B)\sin(4t) &= \sin(4t) \\
-12A - 88B = 1 \quad -88A + 12B &= 0 \\
A = \frac{3}{1972} \quad B &= \frac{11}{986}
\end{aligned}$$

$$\begin{aligned}
y_p &= \frac{3}{1972}\cos(4t) + \frac{11}{986}\sin(4t) \\
amplitude &= \sqrt{(\frac{3}{1972})^2 + (\frac{11}{986})^2} \\
\phi &= \tan^{-1}(\frac{B}{A}) \approx 82.23 \\
T &= \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} \\
f &= \frac{\omega}{2\pi} = \frac{2}{\pi}
\end{aligned}$$

Section 7.2

Exercise 3

Use Definition 1 to determine the Laplace transform of the given function.

$$\begin{aligned}
\mathcal{L}\{e^{6t}\} &= \int_0^\infty e^{-st}e^{6t}\ dt \\
&= \lim_{N\to\infty}\int_0^N e^{6t-st}\ dt \\
&= \lim_{N\to\infty}\int_0^N e^{(6-s)t}\ dt \\
&= \lim_{N\to\infty}\left[\frac{1}{6-s}e^{(6-s)t}\right]_0^N \\
&= 0 - \frac{1}{6-s}, \quad s > 6
\end{aligned}$$

Exercise 11

Use Definition 1 to determine the Laplace transform of the given function.

$$\begin{aligned}
f(t) &= \begin{cases} \sin(t), & 0 < t < 1 \\ 0, & \pi < t \end{cases} \\
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}f(t)\ dt \\
&= \int_0^\pi e^{-st}\sin(t)\ dt + \int_\pi^\infty e^{-st}0\ dt \\
&= \int_0^\pi e^{-st}\sin(t)\ dt \\
&= \left[-\frac{e^{-st}(s\sin(t)+\cos(t))}{s^2+1}\right]_0^\pi \\
&= \frac{-e^{-s\pi}(-1)}{s^2+1} - \frac{-e^0(1)}{s^2+1} \\
&= \frac{e^{-s\pi}+1}{s^2+1}
\end{aligned}$$

Exercise 13

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}
\mathcal{L}\{6e^{-3t} - t^2 + 2t - 8\} &= 6\mathcal{L}\{e^{-3t}\} - \mathcal{L}\{t^2\} + 2\mathcal{L}\{t\} - 8\mathcal{L}\{1\} \\
&= 6(\frac{1}{s+3}) - \frac{2!}{s^3} + 2(\frac{1}{s^2}) - 8(\frac{1}{s}) \\
&= \frac{6}{s+3} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{8}{s}, \quad s > 0
\end{aligned}$$

Exercise 14

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{5-\mathrm{e}^{2t}+6t^2\} &= 5\mathcal{L}\{1\}-\mathcal{L}\{\mathrm{e}^{2t}\}+6\mathcal{L}\{t^2\}\\ &= 5(\frac{1}{s})-\frac{1}{s-2}+6(\frac{2!}{s^3})\\ &= \frac{5}{s}-\frac{1}{s-2}+\frac{12}{s^3}, \quad s>2\end{aligned}$$

Exercise 15

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{t^3-t\mathrm{e}^{t}+\mathrm{e}^{4t}\cos(t)\} &= \mathcal{L}\{t^3\}-\mathcal{L}\{t\mathrm{e}^{t}\}+\mathcal{L}\{\mathrm{e}^{4t}\cos(t)\}\\ &= \frac{3!}{s^4}-\frac{1}{(s-1)^2}+\frac{s-4}{(s-4)^2+1^2}\\ &= \frac{6}{s^4}-\frac{1}{(s-2)^2}+\frac{s-4}{(s-4)^2+1}, \quad s>4\end{aligned}$$

Exercise 16

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{t^2-3t-2\mathrm{e}^{-t}\sin(3t)\} &= \mathcal{L}\{t^2\}-3\mathcal{L}\{t\}-2\mathcal{L}\{\mathrm{e}^{-t}\sin(3t)\}\\ &= \frac{2}{s^3}-3(\frac{1}{s^2})-2(\frac{3}{(s+1)^2+3^2})\\ &= \frac{2}{s^3}-\frac{3}{s^2}-\frac{6}{(s+1)^2+9}, \quad s>0\end{aligned}$$

Exercise 17

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{\mathrm{e}^{3t}\sin(6t)-t^3+\mathrm{e}^{t}\} &= \mathcal{L}\{\mathrm{e}^{3t}\sin(6t)\}-\mathcal{L}\{t^3\}+\mathcal{L}\{\mathrm{e}^{t}\}\\ &= \frac{6}{(s-3)^2+6^2}-\frac{3!}{s^4}+\frac{1}{s-1}\\ &= \frac{6}{(s-3)^2+36}-\frac{6}{s^4}+\frac{1}{s-1}, \quad s>3\end{aligned}$$

Exercise 18

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{t^4 - t^2 - t + \sin(\sqrt{2}t)\} &= \mathcal{L}\{t^4\} - \mathcal{L}\{t^2\} - \mathcal{L}\{t\} + \mathcal{L}\{\sin(\sqrt{2}t)\} \\ &= \frac{4!}{s^5} - \frac{2!}{s^3} - \frac{1!}{s^2} + \frac{\sqrt{2}}{s^2 + (\sqrt{2})^2} \\ &= \frac{24}{s^5} - \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2+2}, \quad s > 0\end{aligned}$$

Exercise 19

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

$$\begin{aligned}\mathcal{L}\{t^4 e^{5t} - e^t \cos(\sqrt{7}t)\} &= \mathcal{L}\{t^4 e^{5t}\} - \mathcal{L}\{e^t \cos(\sqrt{7}t)\} \\ &= \frac{4!}{(s-5)^5} - \frac{s-1}{(s-1)^2 + (\sqrt{7})^2} \\ &= \frac{24}{(s-5)^5} - \frac{s-1}{(s-1)^2+7}, \quad s < 5\end{aligned}$$

Exercise 20

Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

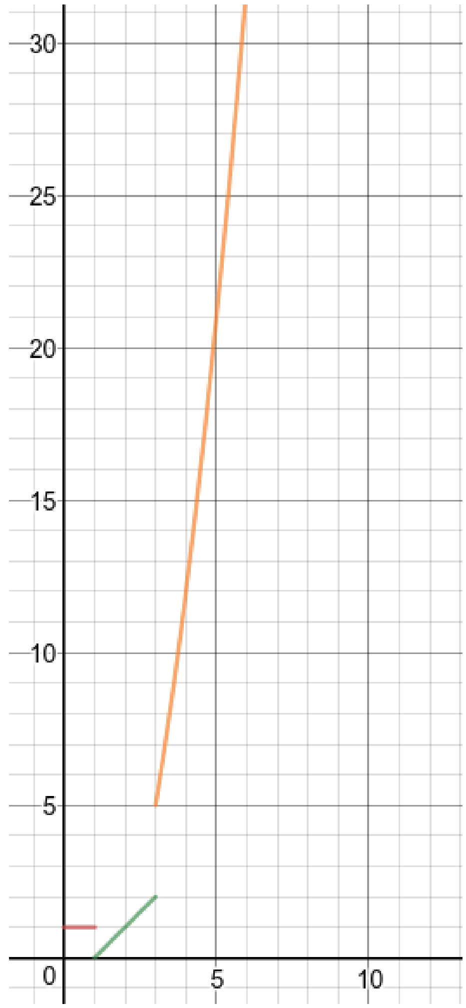
$$\begin{aligned}\mathcal{L}\{e^{-2t} \cos(\sqrt{3}t) - t^2 e^{-2t}\} &= \mathcal{L}\{e^{-2t} \cos(\sqrt{3}t)\} - \mathcal{L}\{t^2 e^{-2t}\} \\ &= \frac{s+2}{(s+2)^2 + (\sqrt{3})^2} - \frac{2!}{(s+2)^3} \\ &= \frac{s+2}{(s+2)^2+3} - \frac{2}{(s+2)^3}, \quad s > -2\end{aligned}$$

Exercise 23

Determine whether $f(t)$ is continuous, piecewise continuous, or neither on $[0, 10]$ and sketch the graph of $f(t)$.

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t-1, & 1 < t < 3 \\ t^2 - 4, & 3 < t \le 10 \end{cases}$$

This function is piecewise continouous.

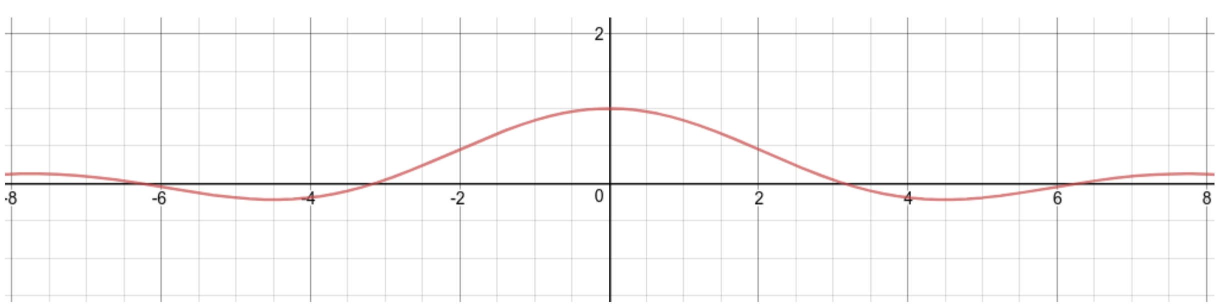


Exercise 28

Determine whether $f(t)$ is continuous, piecewise continuous, or neither on $[0, 10]$ and sketch the graph of $f(t)$.

$$f(t) = \begin{cases} \frac{\sin(t)}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

$f(t)$ is continuous and piecewise continuous.



Exercise 30

For the transforms $F(s)$ in Table 7.1, what can be said about $\lim_{s\to\infty} F(s)$?

s is mostly in the denominator and increases in such a way that as $s \to \infty$, $F(s)$ goes to 0.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech