Differential Equations: Homework 7

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Section 4.4

Exercise 13

Find a particular solution to the differential equation.

$$\begin{aligned}
y'' - y' + 9y &= 3\sin(3t) \\
r^2 - r + 9 &= 0 \\
r &= \frac{1 \pm \sqrt{1-4(9)}}{2} \\
&= \frac{1}{2} \pm \frac{\sqrt{35}}{2} i \\
y_p &= A\sin(3t) + B\cos(3t) \\
y_p' &= 3A\cos(3t) - 3B\sin(3t) \\
y_p'' &= -9A\sin(3t) - 9B\cos(3t)
\end{aligned}$$

$$\begin{aligned}
-9A\sin(3t) - 9B\cos(3t) - (3A\cos(3t) - 3B\sin(3t)) + 9(A\sin(3t) + B\cos(3t)) &= 3\sin(3t) \\
-3A\cos(3t) - 3B\sin(3t) &= 3\sin(3t) \\
-3A = 0 \quad -3B &= 3 \\
A = 0 \quad B &= -1 \\
y_p &= \cos(3t)
\end{aligned}$$

Exercise 16

Find a particular solution to the differential equation.

$$\begin{aligned}
\theta''(t) - \theta(t) &= t\sin(t) \\
r^2 - r &= r(r-1) = 0 \\
r &= 0 \quad r = 1 \\
y_p &= (At+B)\cos(t) + (Ct+D)\sin(t) \\
y_p' &= (At+B)(-\sin(t)) + A\cos(t) + (Ct+D)\cos(t) + C\sin(t) \\
y_p'' &= (At+B)(-\cos(t)) - A\sin(t) - A\sin(t) + (Ct+D)(-\sin(t)) + C\cos(t) + C\cos(t) \\
&= (2C - At - B)\cos(t) - (2A + Ct + D)\sin(t)
\end{aligned}$$

$$\begin{aligned}
(2C - At - B)\cos(t) - (2A + Ct + D)\sin(t) - ((At+B)\cos(t) + (Ct+D)\sin(t)) &= 0 \\
(2C - 2B)\cos(t) + (-2A)t\cos(t) + (-2A - 2D)\sin(t) + (-2C)t\sin(t) &= t\sin(t) \\
2C - 2B = 0 \quad -2A = 0 \quad -2A - 2D = 0 \quad -2C &= 1 \\
A = 0 \quad B = -\frac{1}{2} \quad C = -\frac{1}{2} \quad D &= 0
\end{aligned}$$

$$y_p = -\frac{1}{2}\cos(t) - \frac{1}{2}t\sin(t)$$

Exercise 17

Find a particular solution to the differential equation.

$$\begin{aligned}
y'' + 4y &= 8\sin(2t) \\
r^2 + 4 &= 0 \\
r &= 0 \pm 2i \\
y_p &= At\sin(2t) + Bt\cos(2t) \\
y_p' &= 2At\cos(2t) + A\sin(2t) - 2Bt\sin(2t) + B\cos(2t) \\
&= (2At + B)\cos(2t) + (-2Bt + A)\sin(2t) \\
y_p'' &= (2At + B)(-2\sin(2t)) + 2A\cos(2t) + (-2Bt + A)(2\cos(2t)) - 2B\sin(2t) \\
&= (-4At - 4B)\sin(2t) + (-4Bt + 4A)\cos(2t) \\
8\sin(2t) &= (-4At - 4B)\sin(2t) + (-4Bt + 4A)\cos(2t) + 4(At\sin(2t) + Bt\cos(2t)) \\
&= (-4B)\sin(2t) + (4A)\cos(2t) \\
-4B &= 8 \quad 4A = 0 \\
B &= -2 \quad A = 0 \\
y_p &= -2t\cos(2t)
\end{aligned}$$

Section 4.5

Exercise 1

Given that $y_1(t) = \cos(t)$ is a solution to $y'' - y' + y = \sin(t)$ and $y_2(t) = \mathrm{e}^{\frac{\mathrm{e}^{2t}}{3}}$ is a solution to $y'' - y' + y = \mathrm{e}^{2t}$ use the superposition principle to find solutions to the following differential equations.

1. $y'' - y' + y = 5\sin(t)$

$$\begin{aligned}
y'' - y' + y &= 5\sin(t) \\
y &= 5y_1(t) + 0y_2(t) = 5\cos(t)
\end{aligned}$$

2. $y'' - y' + y = \sin(t) - 3\mathrm{e}^{2t}$

$$\begin{aligned}
y'' - y' + y &= \sin(t) - 3\mathrm{e}^{2t} \\
y &= 1y_1(t) - 3y_2(t) = \cos(t) - 3\mathrm{e}^{\frac{2t}{3}}
\end{aligned}$$

3. $y'' - y' + y = 4\sin(t) + 18\mathrm{e}^{2t}$

$$\begin{aligned}
y'' - y + y &= 4\sin(t) + 18\mathrm{e}^{2t} \\
y &= 4y_t(t) + 18y_2(t) = 4\cos(t) + 18\mathrm{e}^{\frac{2t}{3}}
\end{aligned}$$

Exercise 5

A nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

$$\begin{aligned}
\theta'' - \theta' - 2\theta &= 1 - 2t \quad \theta_p(t) = t - 1 \\
r^2 - r - 2 &= (r - 2)(r + 1) = 0 \\
r &= 2 \quad r = -1 \\
\theta_h(t) &= c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{-t} \\
\theta(t) &= \theta_h(t) + \theta_p(t) \\
&= c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{-t} + t - 1
\end{aligned}$$

Exercise 7

A nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

$$\begin{aligned}
y'' &= 2y + 2\tan^3(x) \quad y_p(x) = \tan(x) \\
y'' - 2y &= 2\tan^3(x) \\
r^2 - 2 &= 0 \quad r = \pm\sqrt{2} \\
y_h(x) &= c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x} \\
y(x) &= y_h(x) + y_p(x) \\
&= c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x} + \tan(x)
\end{aligned}$$

Exercise 17

Find a general solution to the given differential equation.

$$\begin{aligned}
y'' - 2y' - 3y &= 3t^2 - 5 \\
r^2 - 2r - 3 &= (r-3)(r+1) = 0 \\
r &= 3 \quad r = -1 \\
y_h &= c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-t} \\
y_p &= At^2 + Bt + C \\
y_p' &= 2At + B \\
y_p'' &= 2A \\
2A - 2(2At + B) - 3(At^2 + Bt + C) &= 3t^2 - 5 \\
-3At^2 + (-4A - 3B)t + (2A - 2B - 3C) &= 3t^2 - 5 \\
-3A = 3 \quad -4A - 3B = 0 \quad 2A - 2B - 3C &= -5 \\
A = -1 \quad B = \frac{4}{3} \quad C &= \frac{1}{9} \\
y_p &= -t^2 + \frac{4}{3}t + \frac{1}{9} \\
y &= y_h + y_p \\
&= c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-t} - t^2 + \frac{4}{3}t + \frac{1}{9}
\end{aligned}$$

Exercise 19

Find a general solution to the given differential equation.

$$
\begin{aligned}
y''(x) - 3y'(x) + 2y(x) &= \mathrm{e}^x \sin(x) \\
r^2 - 3r + 2 &= (r-2)(r-1) = 0 \\
r &= 2 \quad r = 1 \\
y_h &= c_1\mathrm{e}^{2x} + c_2\mathrm{e}^x \\
y_p &= A\mathrm{e}^x \sin(x) + B\mathrm{e}^x \cos(x) \\
y_p' &= A\mathrm{e}^x \cos(x) + A\mathrm{e}^x \sin(x) - B\mathrm{e}^x \sin(x) + B\mathrm{e}^x \cos(x) \\
&= (A\mathrm{e}^x + B\mathrm{e}^x)\cos(x) + (A\mathrm{e}^x - B\mathrm{e}^x)\sin(x) \\
y_p'' &= (A\mathrm{e}^x + B\mathrm{e}^x)(-\sin(x)) + (A\mathrm{e}^x + B\mathrm{e}^x)\cos(x) + \\
&\quad (A\mathrm{e}^x - B\mathrm{e}^x)\cos(x) + (A\mathrm{e}^x - B\mathrm{e}^x)\sin(x) \\
&= -2B\mathrm{e}^x \sin(x) + 2A\mathrm{e}^x \cos(x) \\
\mathrm{e}^x \sin(x) &= -2B\mathrm{e}^x \sin(x) + 2A\mathrm{e}^x \cos(x) - \\
&\quad 3((A\mathrm{e}^x + B\mathrm{e}^x)\cos(x) + (A\mathrm{e}^x - B\mathrm{e}^x)\sin(x)) + \\
&\quad 2(A\mathrm{e}^x \sin(x) + B\mathrm{e}^x \cos(x)) \\
\sin(x) &= (-2B + 3B - 3A + 2A)\sin(x) + (2A - 3A - 3B + 2B)\cos(x) \\
B - A &= 1 \quad -A - B = 0 \\
A &= -\frac{1}{2} \quad B = \frac{1}{2} \\
y_p &= -\frac{1}{2}\mathrm{e}^x \sin(x) + \frac{1}{2}\mathrm{e}^x \cos(x) \\
y &= y_h + y_p \\
&= c_1\mathrm{e}^{2x} + c_2\mathrm{e}^x - \frac{1}{2}\mathrm{e}^x \sin(x) + \frac{1}{2}\mathrm{e}^x \cos(x)
\end{aligned}
$$

Exercise 25

Find the solution to the initial value problem.

$$
\begin{aligned}
z''(x) + z(x) &= 2\mathrm{e}^{-x} \quad z(0) = 0 \quad z'(0) = 0 \\
r^2 + 1 &= 0 \\
r &= 0 \pm i \\
z_h(x) &= \mathrm{e}^{0x}\left[c_1 \cos(1x) + c_2 \sin(1x)\right] \\
&= c_1 \cos(x) + c_2 \sin(x) \\
z_p(x) &= \mathrm{e}^{-x} \\
z(x) &= z_h(x) + z_p(x) \\
&= c_1 \cos(x) + c_2 \sin(x) + \mathrm{e}^{-x} \\
z(0) &= 0 = c_1 \cos(0) + c_2 \sin(0) + \mathrm{e}^0 \\
c_1 &= -1 \\
z'(x) &= -c_1 \sin(x) + c_2 \cos(x) - \mathrm{e}^{-x} \\
z'(0) &= 0 = -c_1 \sin(0) + c_2 \cos(0) - \mathrm{e}^0 \\
c_2 &= 1 \\
z(x) &= -\cos(x) + \sin(x) + \mathrm{e}^{-x}
\end{aligned}
$$

Exercise 29

Find the solution to the initial value problem.

$$\begin{aligned}
y''(\theta) - y(\theta) &= \sin\theta - \mathrm{e}^{2\theta} \quad y(0) = 1 \quad y'(0) = -1 \\
r^2 - 1 &= 0 \\
r &= \pm 1 \\
y_h(\theta) &= c_1\mathrm{e}^{\theta} + c_2\mathrm{e}^{-\theta} \\
y_p(\theta) &= A\sin\theta + B\cos\theta + C\mathrm{e}^{2\theta} \\
y_p'(\theta) &= A\cos\theta - B\sin\theta + 2C\mathrm{e}^{2\theta} \\
y_p''(\theta) &= -A\sin\theta - B\cos\theta + 4C\mathrm{e}^{2\theta} \\
\sin\theta - \mathrm{e}^{2\theta} &= -A\sin\theta - B\cos\theta + 4C\mathrm{e}^{2\theta} - (A\sin\theta + B\cos\theta + C\mathrm{e}^{2\theta}) \\
\sin\theta - \mathrm{e}^{2\theta} &= (-2A)\sin\theta + (-2B)\cos\theta + (3C)\mathrm{e}^{2\theta} \\
-2A &= 1 \quad -2B = 0 \quad 3C = -1 \\
A &= -\frac{1}{2} \quad B = 0 \quad C = -\frac{1}{3} \\
y_p(\theta) &= -\frac{1}{2}\sin\theta - \frac{1}{3}\mathrm{e}^{2\theta} \\
y(\theta) &= y_h(\theta) + y_p(\theta) \\
&= c_1\mathrm{e}^{\theta} + c_2\mathrm{e}^{-\theta} - \frac{\sin\theta}{2} - \frac{\mathrm{e}^{2\theta}}{3} \\
y(0) = 1 &= c_1\mathrm{e}^{0} + c_2\mathrm{e}^{0} - \frac{\sin(0)}{2} - \frac{\mathrm{e}^{0}}{3} \\
1 &= c_1 + c_2 - \frac{1}{3} \\
y'(\theta) &= c_1\mathrm{e}^{\theta} - c_2\mathrm{e}^{-\theta} - \frac{\cos\theta}{2} - \frac{2\mathrm{e}^{2\theta}}{3} \\
y'(0) = -1 &= c_1\mathrm{e}^{0} - c_2\mathrm{e}^{0} - \frac{\cos 0}{2} - \frac{2\mathrm{e}^{0}}{3} \\
-1 &= c_1 - c_2 - \frac{1}{2} - \frac{2}{3} \\
c_1 &= \frac{3}{4} \quad c_2 = \frac{7}{12} \\
y(\theta) &= \frac{3}{4}\mathrm{e}^{\theta} + \frac{7}{12}\mathrm{e}^{-\theta} - \frac{\sin\theta}{2} - \frac{\mathrm{e}^{2\theta}}{3}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech