

Differential Equations: Homework 7

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Section 4.4

Exercise 13

Find a particular solution to the differential equation.

$$y'' - y' + 9y = 3 \sin(3t)$$
$$r^2 - r + 9 = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4(9)}}{2}$$
$$= \frac{1}{2} \pm \frac{\sqrt{35}}{2}i$$

$$y_p = A \sin(3t) + B \cos(3t)$$
$$y'_p = 3A \cos(3t) - 3B \sin(3t)$$
$$y''_p = -9A \sin(3t) - 9B \cos(3t)$$

$$-9A \sin(3t) - 9B \cos(3t) - (3A \cos(3t) - 3B \sin(3t)) + 9(A \sin(3t) + B \cos(3t)) = 3 \sin(3t)$$
$$-3A \cos(3t) - 3B \sin(3t) = 3 \sin(3t)$$
$$-3A = 0 \quad -3B = 3$$
$$A = 0 \quad B = -1$$
$$y_p = \cos(3t)$$

Exercise 16

Find a particular solution to the differential equation.

$$\theta''(t) - \theta(t) = t \sin(t)$$

$$r^2 - r = r(r - 1) = 0$$

$$r = 0 \quad r = 1$$

$$y_p = (At + B) \cos(t) + (Ct + D) \sin(t)$$

$$y'_p = (At + B)(-\sin(t)) + A \cos(t) + (Ct + D) \cos(t) + C \sin(t)$$

$$y''_p = (At + B)(-\cos(t)) - A \sin(t) - A \sin(t) + (Ct + D)(-\sin(t)) + C \cos(t) + C \cos(t)$$
$$= (2C - At - B) \cos(t) - (2A + Ct + D) \sin(t)$$

$$(2C - At - B) \cos(t) - (2A + Ct + D) \sin(t) - ((At + B) \cos(t) + (Ct + D) \sin(t)) = 0$$
$$(2C - 2B) \cos(t) + (-2A)t \cos(t) + (-2A - 2D) \sin(t) + (-2C)t \sin(t) = t \sin(t)$$
$$2C - 2B = 0 \quad -2A = 0 \quad -2A - 2D = 0 \quad -2C = 1$$
$$A = 0 \quad B = -\frac{1}{2} \quad C = -\frac{1}{2} \quad D = 0$$

$$y_p = -\frac{1}{2} \cos(t) - \frac{1}{2}t \sin(t)$$

Exercise 17

Find a particular solution to the differential equation.

$$y'' + 4y = 8 \sin(2t)$$

$$r^2 + 4 = 0$$

$$r = 0 \pm 2i$$

$$y_p = At \sin(2t) + Bt \cos(2t)$$

$$y'_p = 2At \cos(2t) + A \sin(2t) - 2Bt \sin(2t) + B \cos(2t)$$

$$= (2At + B) \cos(2t) + (-2Bt + A) \sin(2t)$$

$$y''_p = (2At + B)(-2 \sin(2t)) + 2A \cos(2t) + (-2Bt + A)(2 \cos(2t)) - 2B \sin(2t)$$

$$= (-4At - 4B) \sin(2t) + (-4Bt + 4A) \cos(2t)$$

$$8 \sin(2t) = (-4At - 4B) \sin(2t) + (-4Bt + 4A) \cos(2t) + 4(At \sin(2t) + Bt \cos(2t))$$

$$= (-4B) \sin(2t) + (4A) \cos(2t)$$

$$-4B = 8 \quad 4A = 0$$

$$B = -2 \quad A = 0$$

$$y_p = -2t \cos(2t)$$

Section 4.5

Exercise 1

Given that $y_1(t) = \cos(t)$ is a solution to $y'' - y' + y = \sin(t)$ and $y_2(t) = e^{\frac{2t}{3}}$ is a solution to $y'' - y' + y = e^{2t}$ use the superposition principle to find solutions to the following differential equations.

1. $y'' - y' + y = 5 \sin(t)$

$$y'' - y' + y = 5 \sin(t)$$

$$y = 5y_1(t) + 0y_2(t) = 5 \cos(t)$$

2. $y'' - y' + y = \sin(t) - 3e^{2t}$

$$y'' - y' + y = \sin(t) - 3e^{2t}$$

$$y = 1y_1(t) - 3y_2(t) = \cos(t) - 3e^{\frac{2t}{3}}$$

3. $y'' - y' + y = 4 \sin(t) + 18e^{2t}$

$$y'' - y' + y = 4 \sin(t) + 18e^{2t}$$

$$y = 4y_1(t) + 18y_2(t) = 4 \cos(t) + 18e^{\frac{2t}{3}}$$

Exercise 5

A nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

$$\theta'' - \theta' - 2\theta = 1 - 2t \quad \theta_p(t) = t - 1$$

$$r^2 - r - 2 = (r - 2)(r + 1) = 0$$

$$r = 2 \quad r = -1$$

$$\theta_h(t) = c_1 e^{2t} + c_2 e^{-t}$$

$$\theta(t) = \theta_h(t) + \theta_p(t)$$

$$= c_1 e^{2t} + c_2 e^{-t} + t - 1$$

Exercise 7

A nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

$$\begin{aligned}y'' &= 2y + 2 \tan^3(x) & y_p(x) &= \tan(x) \\y'' - 2y &= 2 \tan^3(x) \\r^2 - 2 &= 0 & r &= \pm\sqrt{2} \\y_h(x) &= c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} \\y(x) &= y_h(x) + y_p(x) \\&= c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + \tan(x)\end{aligned}$$

Exercise 17

Find a general solution to the given differential equation.

$$\begin{aligned}y'' - 2y' - 3y &= 3t^2 - 5 \\r^2 - 2r - 3 &= (r - 3)(r + 1) = 0 \\r &= 3 & r &= -1 \\y_h &= c_1 e^{3t} + c_2 e^{-t} \\y_p &= At^2 + Bt + C \\y'_p &= 2At + B \\y''_p &= 2A \\2A - 2(2At + B) - 3(At^2 + Bt + C) &= 3t^2 - 5 \\-3At^2 + (-4A - 3B)t + (2A - 2B - 3C) &= 3t^2 - 5 \\-3A = 3 & \quad -4A - 3B = 0 & \quad 2A - 2B - 3C = -5 \\A = -1 & \quad B = \frac{4}{3} & \quad C = \frac{1}{9} \\y_p &= -t^2 + \frac{4}{3}t + \frac{1}{9} \\y &= y_h + y_p \\&= c_1 e^{3t} + c_2 e^{-t} - t^2 + \frac{4}{3}t + \frac{1}{9}\end{aligned}$$

Exercise 19

Find a general solution to the given differential equation.

$$y''(x) - 3y'(x) + 2y(x) = e^x \sin(x)$$

$$r^2 - 3r + 2 = (r - 2)(r - 1) = 0$$

$$r = 2 \quad r = 1$$

$$y_h = c_1 e^{2x} + c_2 e^x$$

$$y_p = Ae^x \sin(x) + Be^x \cos(x)$$

$$y'_p = Ae^x \cos(x) + Ae^x \sin(x) - Be^x \sin(x) + Be^x \cos(x)$$

$$= (Ae^x + Be^x) \cos(x) + (Ae^x - Be^x) \sin(x)$$

$$y''_p = (Ae^x + Be^x)(-\sin(x)) + (Ae^x + Be^x) \cos(x) +$$

$$(Ae^x - Be^x) \cos(x) + (Ae^x - Be^x) \sin(x)$$

$$= -2Be^x \sin(x) + 2Ae^x \cos(x)$$

$$e^x \sin(x) = -2Be^x \sin(x) + 2Ae^x \cos(x) -$$

$$3((Ae^x + Be^x) \cos(x) + (Ae^x - Be^x) \sin(x)) +$$

$$2(Ae^x \sin(x) + Be^x \cos(x))$$

$$\sin(x) = (-2B + 3B - 3A + 2A) \sin(x) + (2A - 3A - 3B + 2B) \cos(x)$$

$$B - A = 1 \quad -A - B = 0$$

$$A = -\frac{1}{2} \quad B = \frac{1}{2}$$

$$y_p = -\frac{1}{2}e^x \sin(x) + \frac{1}{2}e^x \cos(x)$$

$$y = y_h + y_p$$

$$= c_1 e^{2x} + c_2 e^x - \frac{1}{2}e^x \sin(x) + \frac{1}{2}e^x \cos(x)$$

Exercise 25

Find the solution to the initial value problem.

$$z''(x) + z(x) = 2e^{-x} \quad z(0) = 0 \quad z'(0) = 0$$

$$r^2 + 1 = 0$$

$$r = 0 \pm i$$

$$z_h(x) = e^{0x} \left[c_1 \cos(1x) + c_2 \sin(1x) \right]$$

$$= c_1 \cos(x) + c_2 \sin(x)$$

$$z_p(x) = e^{-x}$$

$$z(x) = z_h(x) + z_p(x)$$

$$= c_1 \cos(x) + c_2 \sin(x) + e^{-x}$$

$$z(0) = 0 = c_1 \cos(0) + c_2 \sin(0) + e^0$$

$$c_1 = -1$$

$$z'(x) = -c_1 \sin(x) + c_2 \cos(x) - e^{-x}$$

$$z'(0) = 0 = -c_1 \sin(0) + c_2 \cos(0) - e^0$$

$$c_2 = 1$$

$$z(x) = -\cos(x) + \sin(x) + e^{-x}$$

Exercise 29

Find the solution to the initial value problem.

$$y''(\theta) - y(\theta) = \sin \theta - e^{2\theta} \quad y(0) = 1 \quad y'(0) = -1$$

$$r^2 - 1 = 0$$

$$r = \pm 1$$

$$y_h(\theta) = c_1 e^\theta + c_2 e^{-\theta}$$

$$y_p(\theta) = A \sin \theta + B \cos \theta + C e^{2\theta}$$

$$y'_p(\theta) = A \cos \theta - B \sin \theta + 2C e^{2\theta}$$

$$y''_p(\theta) = -A \sin \theta - B \cos \theta + 4C e^{2\theta}$$

$$\sin \theta - e^{2\theta} = -A \sin \theta - B \cos \theta + 4C e^{2\theta} - (A \sin \theta + B \cos \theta + C e^{2\theta})$$

$$\sin \theta - e^{2\theta} = (-2A) \sin \theta + (-2B) \cos \theta + (3C) e^{2\theta}$$

$$-2A = 1 \quad -2B = 0 \quad 3C = -1$$

$$A = -\frac{1}{2} \quad B = 0 \quad C = -\frac{1}{3}$$

$$y_p(\theta) = -\frac{1}{2} \sin \theta - \frac{1}{3} e^{2\theta}$$

$$y(\theta) = y_h(\theta) + y_p(\theta)$$

$$= c_1 e^\theta + c_2 e^{-\theta} - \frac{\sin \theta}{2} - \frac{e^{2\theta}}{3}$$

$$y(0) = 1 = c_1 e^0 + c_2 e^0 - \frac{\sin(0)}{2} - \frac{e^0}{3}$$

$$1 = c_1 + c_2 - \frac{1}{3}$$

$$y'(\theta) = c_1 e^\theta - c_2 e^{-\theta} - \frac{\cos \theta}{2} - \frac{2e^{2\theta}}{3}$$

$$y'(0) = -1 = c_1 e^0 - c_2 e^0 - \frac{\cos 0}{2} - \frac{2e^0}{3}$$

$$-1 = c_1 - c_2 - \frac{1}{2} - \frac{2}{3}$$

$$c_1 = \frac{3}{4} \quad c_2 = \frac{7}{12}$$

$$y(\theta) = \frac{3}{4} e^\theta + \frac{7}{12} e^{-\theta} - \frac{\sin \theta}{2} - \frac{e^{2\theta}}{3}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech