

Differential Equations: Homework 6

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Section 4.3

Exercise 1

In the problem the auxiliary equation for the general differential equation has complex roots. Find a general solution.

$$\begin{aligned}y'' + 9y &= 0 \\y &= e^{rt} \\y' &= re^{rt} \\y'' &= r^2e^{rt} \\r^2e^{rt} + 9e^{rt} &= 0 \\r^2 + 9 &= 0 \\r &= \frac{0 \pm \sqrt{0 - 4(9)}}{2} \\&= 0 \pm 3i \quad \alpha = 0 \quad \beta = 3 \\y &= e^{0t} \left[c_1 \cos(3t) + c_2 \sin(3t) \right] \\&= c_1 \cos(3t) + c_2 \sin(3t)\end{aligned}$$

Exercise 2

In the problem the auxiliary equation for the general differential equation has complex roots. Find a general solution.

$$\begin{aligned}y'' + y &= 0 \\r^2 + 1 &= 0 \\r &= 0 \pm i \quad \alpha = 0 \quad \beta = 1 \\y &= e^{0t} \left[c_1 \cos(t) + c_2 \sin(t) \right] \\y &= c_1 \cos(t) + c_2 \sin(t)\end{aligned}$$

Exercise 3

In the problem the auxiliary equation for the general differential equation has complex roots. Find a general solution.

$$\begin{aligned}z'' - 6z' + 10z &= 0 \\r^2 - 6r + 10 &= 0 \\r &= \frac{6 \pm \sqrt{36 - 4(10)}}{2} \\&= \frac{6 \pm \sqrt{-4}}{2} \\&= 3 \pm i \quad \alpha = 3 \quad \beta = 1 \\z &= e^{3t} \left[c_1 \cos(t) + c_2 \sin(t) \right]\end{aligned}$$

Exercise 11

Find a general solution.

$$\begin{aligned}
z'' + 10z' + 25z &= 0 \\
r^2 + 10r + 25 &= 0 \\
r &= 5 \\
z &= c_1 e^{5t} + c_2 t e^{5t}
\end{aligned}$$

Exercise 13

Find a general solution.

$$\begin{aligned}
y'' - 2y' + 26y &= 0 \\
r^2 - 2r + 26 &= 0 \\
r &= \frac{4 \pm \sqrt{16 - 4(26)}}{2} \\
&= 2 \pm 22i \quad \alpha = 2 \quad \beta = \sqrt{22} \\
y &= e^{2t} \left[c_1 \cos(\sqrt{22}t) + c_2 \sin(\sqrt{22}t) \right]
\end{aligned}$$

Exercise 21

Solve the given initial value problem.

$$\begin{aligned}
y'' + 2y' + 2y &= 0 \\
r^2 + 2r + 2 &= 0 \\
r &= \frac{-2 \pm \sqrt{4 - 4(2)}}{2} \\
&= -1 \pm i \quad \alpha = -1 \quad \beta = 1 \\
y &= e^{-t} \left[c_1 \cos(t) + c_2 \sin(t) \right]
\end{aligned}$$

Given the initial values of $y(0) = 2$ and $y'(0) = 1$:

$$\begin{aligned}
y(t) &= e^{-t} \left[c_1 \cos(t) + c_2 \sin(t) \right] \\
y(0) = 2 &= e^0 \left[c_1 \cos(0) + c_2 \sin(0) \right] \\
c_1 &= 2 \\
y'(t) &= e^{-t} \left[-c_1 \sin(t) + c_2 \cos(t) \right] - e^{-t} \left[c_1 \cos(t) + c_2 \sin(t) \right] \\
y'(0) = 1 &= e^0 \left[-c_1 \sin(0) + c_2 \cos(0) \right] - e^0 \left[c_1 \cos(0) + c_2 \sin(0) \right] \\
&= c_2 - c_1 \\
c_2 &= 3 \\
y(t) &= e^{-t} \left[2 \cos(t) + 3 \sin(t) \right]
\end{aligned}$$

Exercise 22

Solve the given initial value problem.

$$y'' + 2y' + 17y = 0$$

$$r^2 + 2r + 17 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(17)}}{2}$$

$$= -1 \pm 4i \quad \alpha = -1 \quad \beta = 4$$

$$y = e^{-t} \left[c_1 \cos(4t) + c_2 \sin(4t) \right]$$

Given the initial values of $y(0) = 1$ and $y'(0) = -1$:

$$y(t) = e^{-t} \left[c_1 \cos(4t) + c_2 \sin(4t) \right]$$

$$y(0) = 1 = c_1$$

$$y'(t) = e^{-t} \left[-4c_1 \sin(4t) + 4c_2 \cos(4t) \right] - e^{-t} \left[c_1 \cos(4t) + c_2 \sin(4t) \right]$$

$$y'(0) = -1 = 4c_2 - c_1$$

$$c_2 = 0$$

$$y(t) = e^{-t} \cos(4t)$$

Exercise 23

Solve the given initial value problem.

$$w'' - 4w' + 2w = 0$$

$$r^2 - 4r + 2 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 4(2)}}{2}$$

$$= 2 \pm \sqrt{2}$$

$$w = c_1 e^{(2+\sqrt{2})t} + c_2 e^{(2-\sqrt{2})t}$$

Given the initial values of $w(0) = 0$ and $w'(0) = 1$:

$$w(t) = c_1 e^{(2+\sqrt{2})t} + c_2 e^{(2-\sqrt{2})t}$$

$$w(0) = 0 = c_1 + c_2$$

$$w'(t) = (2 + \sqrt{2})c_1 e^{(2+\sqrt{2})t} + (2 - \sqrt{2})c_2 e^{(2-\sqrt{2})t}$$

$$w'(0) = 1 = (2 + \sqrt{2})c_1 + (2 - \sqrt{2})c_2$$

$$c_1 = \frac{\sqrt{2}}{4} \quad c_2 = -\frac{\sqrt{2}}{4}$$

$$w(t) = \frac{\sqrt{2}}{4} e^{(2+\sqrt{2})t} - \frac{\sqrt{2}}{4} e^{(2-\sqrt{2})t}$$

$$= \frac{\sqrt{2}}{4} \left[e^{(2+\sqrt{2})t} - e^{(2-\sqrt{2})t} \right]$$

Exercise 24

Solve the given initial value problem.

$$\begin{aligned}
 y'' + 9y &= 0 \\
 r^2 + 9 &= 0 \\
 r &= 0 \pm 3i \quad \alpha = 0 \quad \beta = 3 \\
 y &= e^0 \left[c_1 \cos(3t) + c_2 \sin(3t) \right]
 \end{aligned}$$

Given the initial values of $y(0) = 1$ and $y'(0) = 1$:

$$\begin{aligned}
 y(t) &= c_1 \cos(3t) + c_2 \sin(3t) \\
 y(0) &= 1 = c_1 \\
 y'(t) &= -3c_1 \sin(3t) + 3c_2 \cos(3t) \\
 y'(0) &= 1 = 3c_2 \\
 c_2 &= \frac{1}{3} \\
 y(t) &= \cos(3t) + \frac{\sin(3t)}{3}
 \end{aligned}$$

Exercise 26

Solve the given initial value problem.

$$\begin{aligned}
 y'' - 2y' + y &= 0 \\
 r^2 - 2r + 1 &= 0 \\
 (r - 1)^2 &= 0 \\
 r &= 1 \\
 y &= c_1 e^t + c_2 t e^t
 \end{aligned}$$

Given the initial values of $y(0) = 1$ and $y'(0) = -2$:

$$\begin{aligned}
 y(t) &= c_1 e^t + c_2 t e^t \\
 y(0) &= 1 = c_1 \\
 y'(t) &= c_1 e^t + c_2 \left[e^t + t e^t \right] \\
 y'(0) &= -2 = c_1 + c_2 \\
 c_2 &= -3 \\
 y(t) &= e^t - 3t e^t
 \end{aligned}$$

Exercise 27

Solve the given initial value problem.

$$y''' - 4y'' + 7y' - 6y = 0$$

$$r^3 - 4r^2 + 7r - 6 = 0$$

$$(r - 2)(r^2 - 2r + 3) = 0$$

$$r = 2$$

$$r = \frac{2 \pm \sqrt{4 - 4(3)}}{2}$$

$$= 1 \pm \sqrt{2}i \quad \alpha = 1 \quad \beta = \sqrt{2}$$

$$y = c_1 e^{2t} + e^t \left[c_2 \cos(\sqrt{2}t) + c_3 \sin(\sqrt{2}t) \right]$$

Given the initial values of $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 0$:

$$y(t) = c_1 e^{2t} + e^t \left[c_2 \cos(\sqrt{2}t) + c_3 \sin(\sqrt{2}t) \right]$$

$$y(0) = 1 = c_1 + c_2$$

$$y'(t) = 2c_1 e^{2t} + e^t \left[-\sqrt{2}c_2 \sin(\sqrt{2}t) + \sqrt{2}c_3 \cos(\sqrt{2}t) \right] + e^t \left[c_2 \cos(\sqrt{2}t) + c_3 \sin(\sqrt{2}t) \right]$$

$$y'(0) = 0 = 2c_1 + \sqrt{2}c_3 + c_2$$

$$y''(t) = 4c_1 e^{2t} + e^t \left[-2c_2 \cos(\sqrt{2}t) - 2c_2 \sin(\sqrt{2}t) \right] + 2e^t \left[-\sqrt{2}c_2 \sin(\sqrt{2}t) + \sqrt{2}c_3 \cos(\sqrt{2}t) \right] + e^t \left[c_2 \cos(\sqrt{2}t) + c_3 \sin(\sqrt{2}t) \right]$$

$$y''(0) = 0 = 4c_1 - 2c_2 + 2\sqrt{2}c_3 + c_2$$

$$c_1 = 1 \quad c_2 = 0 \quad c_3 = -\sqrt{2}$$

$$y(t) = e^{2t} - e^t \sqrt{2} \sin(\sqrt{2}t)$$

Exercise 28

To see the effect of changing the parameter b in the initial value problem

$$y'' + by' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 0$$

solve the problem for $b = 5, 4, 2$ and sketch the solutions. $b = 5$:

$$y'' + 5y' + 4y = 0$$

$$r^2 + 5r + 4 = (r + 4)(r + 1) = 0$$

$$r = -4 \quad r = -1$$

$$y(t) = c_1 e^{-4t} + c_2 e^{-t}$$

$$y(0) = 1 = c_1 + c_2$$

$$y'(t) = -4c_1 e^{-4t} - c_2 e^{-t}$$

$$y'(0) = 0 = -4c_1 - c_2$$

$$c_1 = -\frac{1}{3} \quad c_2 = \frac{4}{3}$$

$$y = -\frac{1}{3} e^{-4t} + \frac{4}{3} e^{-t}$$

$b = 4$:

$$y'' + 4y' + 4y = 0$$

$$r^2 + 4r + 4 = (r + 2)^2 = 0$$

$$r = -2$$

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$y(0) = 1 = c_1$$

$$y'(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}$$

$$y'(0) = 0 = -2c_1 - c_2$$

$$c_2 = -2$$

$$y(t) = e^{-2t} - 2t e^{-2t}$$

$b = 2$:

$$y'' + 2y' + 4y = 0$$

$$r^2 + 2r + 4 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(4)}}{2}$$

$$= -1 \pm \sqrt{3}i \quad \alpha = -1 \quad \beta = \sqrt{3}$$

$$y(t) = e^{-t} \left[c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t) \right]$$

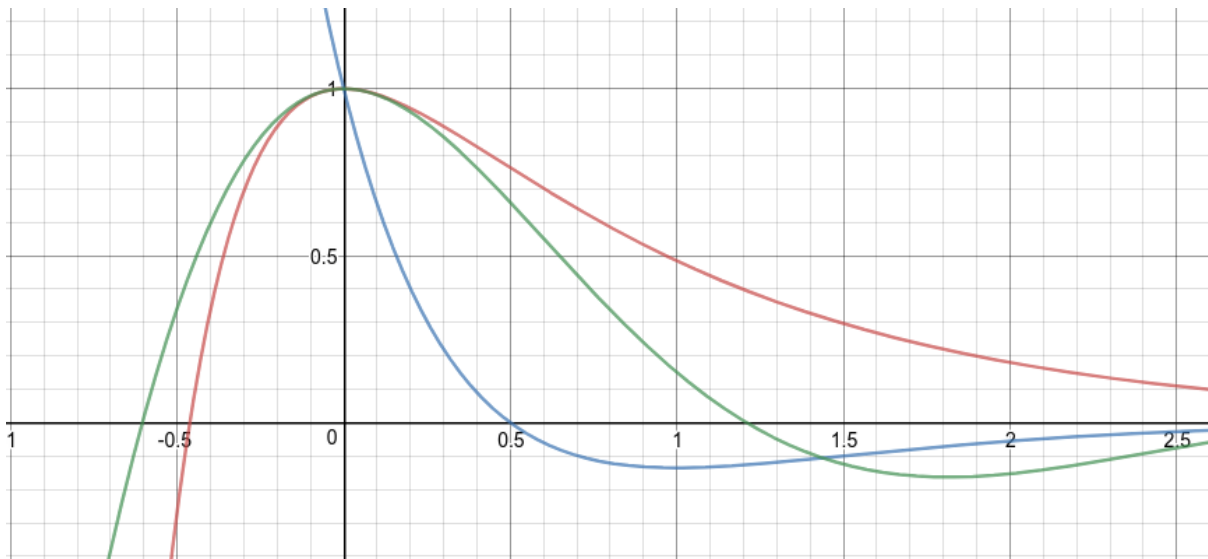
$$y(0) = 1 = c_1$$

$$y'(t) = e^{-t} \left[-\sqrt{3}c_1 \sin(\sqrt{3}t) + \sqrt{3}c_2 \cos(\sqrt{3}t) \right] + e^{-t} \left[c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t) \right]$$

$$y'(0) = 0 = -\sqrt{3}c_2 + c_1$$

$$c_2 = \frac{1}{\sqrt{3}}$$

$$y(t) = e^{-t} \left[\cos(\sqrt{3}t) + \frac{\sin(\sqrt{3}t)}{\sqrt{3}} \right]$$



Section 4.4

Exercise 1

Decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

$$y'' + 2y' - y = t^{-1}e^t$$

$\frac{e^t}{t}$ cannot be formed as $Ct^m e^{rt}$, so the method of undetermined coefficients cannot be applied.

Exercise 4

Decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

$$x'' + 5x' - 3x = 3^t$$

3^t can be rewritten as $e^{\ln(3)t}$, so the method of undetermined coefficients can be applied.

Exercise 5

Decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

$$y''(\theta) + 3y'(\theta) - y(\theta) = \sec(\theta)$$

$\sec \theta = \frac{1}{\cos \theta}$ cannot be expressed as a product of polynomials, exponentials, and sin, cos, so the method of undetermined coefficients cannot be applied.

Exercise 9

Find a particular solution to the differential equation.

$$\begin{aligned}y'' + 3y &= -9 \\y_p &= A \\y_p'' &= 0 \\0 + 3A &= -9 \\y_p = A &= -3\end{aligned}$$

Exercise 10

Find a particular solution to the differential equation.

$$\begin{aligned}y'' + 2y' - y &= 10 \\y_p &= A \\y_p' &= 0 \\y_p'' &= 0 \\0 - 0 - A &= 10 \\y_p = A &= -10\end{aligned}$$

Exercise 12

Find a particular solution to the differential equation.

$$\begin{aligned}2x' + x &= 3t^2 \\x_p &= At^2 + Bt + C \\x'_p &= 2At + B \\2(2At + B) + At^2 + Bt + C &= 3t^2 \\4At + 2B + At^2 + Bt + C &= 3t^2 \\At^2 + (4A + B)t + (2B + C) &= 3t^2 \\A = 3 \quad 4A + B = 0 \quad 2B + C = 0 \\A = 3 \quad B = -12 \quad C = 24 \\x_p &= 3t^2 - 12t + 24\end{aligned}$$

Exercise 14

Find a particular solution to the differential equation.

$$\begin{aligned}2z'' + z &= 9e^{2t} \\z_p &= Ae^{2t} \\z'_p &= 2Ae^{2t} \\z''_p &= 4Ae^{2t} \\2(4Ae^{2t}) + Ae^{2t} &= 9e^{2t} \\8A + A &= 9 \\A &= 1 \\z_p &= e^{2t}\end{aligned}$$

Exercise 18

Find a particular solution to the differential equation.

$$\begin{aligned}y'' - 2y' + y &= 8e^t \\y_p = y'_p = y''_p &= Ae^t \\Ae^t - 2(Ae^t) + Ae^t &= 8e^t \\0 &= 8e^t\end{aligned}$$

This does not work since the auxiliary equation $r^2 - 2r + 1$ has the double root $r = 1$.

$$\begin{aligned}y_p &= At^2e^t \\y'_p &= At^2e^t + 2Ate^t = e^t(At^2 + 2At) \\y''_p &= At^2e^t + 2Ate^t + 2Ate^t + 2Ae^t \\&= e^t(At^2 + 4At + 2A) \\e^t(At^2 + 4At + 2A) - 2(At^2e^t + 2Ate^t) + At^2e^t &= 8e^t \\At^2 + 4At + 2A - 2At^2 - 4At + At^2 &= 8 \\2A &= 8 \\A &= 4 \\y_p &= 4t^2e^t\end{aligned}$$

Exercise 23

Find a particular solution to the differential equation.

$$\begin{aligned}y''(\theta) - 7y'(\theta) &= \theta^2 \\r^2 - 7r &= 0 \\r = 0 \quad r = 7 \\y''(\theta) - 7y'(\theta) &= 1\theta^2 e^{0t} \\y_p &= \theta(A\theta^2 + B\theta + C) \\y'_p &= 3A\theta^2 + 2B\theta + C \\y''_p &= 6A\theta + 2B \\6A\theta + 2B - 7(3A\theta^2 + 2B\theta + C) &= \theta^2 \\-21A\theta^2 + (6A - 14B)\theta + (6A + 2B - 7C) &= \theta^2 \\-21A = 1 \quad 6A - 14B = 0 \quad 2B - 7C = 0 \\A = -\frac{1}{21} \quad B = -\frac{1}{49} \quad C = -\frac{2}{343} \\y_p &= -\frac{\theta^3}{21} - \frac{\theta^2}{49} - \frac{2\theta}{343}\end{aligned}$$

Exercise 28

Determine the form of a particular solution for the differential equation.

$$\begin{aligned}y'' - 6y' + 9y &= 5t^6 e^{3t} \\C = 5 \quad m = 6 \quad r = 3 \\r^2 - 6r + 9 &= 0 \\(r - 3)^3 &= 0 \\r &= 3 \\y_p(t) &= t^2(At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft + G)e^{rt}\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech