Differential Equations: Homework 6

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Section 4.3

Exercise 1

In the problem the auxiliary equation for the general differential equation has complex roots. Find a general solution.

$$\begin{aligned}
y'' + 9y &= 0 \\
y &= \mathrm{e}^{rt} \\
y' &= r\mathrm{e}^{rt} \\
y'' &= r^2\mathrm{e}^{rt} \\
r^2\mathrm{e}^{rt} + 9\mathrm{e}^{rt} &= 0 \\
r^2 + 9 &= 0 \\
r &= \frac{0 \pm \sqrt{0 - 4(9)}}{2} \\
&= 0 \pm 3i \quad \alpha = 0 \quad \beta = 3 \\
y &= \mathrm{e}^{0t}\Big[c_1\cos(3t) + c_2\sin(3t)\Big] \\
&= c_1\cos(3t) + c_2\sin(3t)
\end{aligned}$$

Exercise 2

In the problem the auxiliary equation for the general differential equation has complex roots. Find a general solution.

$$\begin{aligned}
y'' + y &= 0 \\
r^2 + 1 &= 0 \\
r &= 0 \pm i \quad \alpha = 0 \quad \beta = 1 \\
y &= \mathrm{e}^{0t}\Big[c_1\cos(t) + c_2\sin(t)\Big] \\
y &= c_1\cos(t) + c_2\sin(t)
\end{aligned}$$

Exercise 3

In the problem the auxiliary equation for the general differential equation has complex roots. Find a general solution.

$$\begin{aligned}
z'' - 6z' + 10z &= 0 \\
r^2 - 6t + 10 &= 0 \\
r &= \frac{6 \pm \sqrt{36 - 4(10)}}{2} \\
&= \frac{6 \pm \sqrt{-4}}{2} \\
&= 3 \pm i \quad \alpha = 3 \quad \beta = 1 \\
z &= \mathrm{e}^{3t}\Big[c_1\cos(t) + c_2\sin(t)\Big]
\end{aligned}$$

Exercise 11

Find a general solution.

$$\begin{aligned} z'' + 10z' + 25z &= 0 \\ r^2 + 10r + 25 &= 0 \\ r &= 5 \\ z &= c_1 \mathrm{e}^{5t} + c_2 t \mathrm{e}^{5t} \end{aligned}$$

Exercise 13

Find a general solution.

$$\begin{aligned} y'' - 2y' + 26y &= 0 \\ r^2 - 2r + 26 &= 0 \\ r &= \frac{4 \pm \sqrt{16 - 4(26)}}{2} \\ &= 2 \pm 22i \quad \alpha = 2 \quad \beta = \sqrt{22} \\ y &= \mathrm{e}^{2t}\Big[c_1 \cos(\sqrt{22}t) + c_2 \sin(\sqrt{22}t)\Big] \end{aligned}$$

Exercise 21

Solve the given initial value problem.

$$\begin{aligned} y'' + 2y' + 2y &= 0 \\ r^2 + 2r + 2 &= 0 \\ r &= \frac{-2 \pm \sqrt{4 - 4(2)}}{2} \\ &= -1 \pm i \quad \alpha = -1 \quad \beta = 1 \\ y &= \mathrm{e}^{-t}\Big[c_1 \cos(t) + c_2 \sin(t)\Big] \end{aligned}$$

Given the initial values of $y(0) = 2$ and $y'(0) = 1$:

$$\begin{aligned} y(t) &= \mathrm{e}^{-t}\Big[c_1 \cos(t) + c_2 \sin(t)\Big] \\ y(0) = 2 &= \mathrm{e}^{0}\Big[c_1 \cos(0) + c_2 \sin(0)\Big] \\ c_1 &= 2 \\ y'(t) &= \mathrm{e}^{-t}\Big[-c_1 \sin(t) + c_2 \cos(t)\Big] - \mathrm{e}^{-t}\Big[c_1 \cos(t) + c_2 \sin(t)\Big] \\ y'(0) = 1 &= \mathrm{e}^{0}\Big[-c_1 \sin(0) + c_2 \cos(0)\Big] - \mathrm{e}^{0}\Big[c_1 \cos(0) + c_2 \sin(0)\Big] \\ &= c_2 - c_1 \\ c_2 &= 3 \\ y(t) &= \mathrm{e}^{-t}\Big[2 \cos(t) + 3 \sin(t)\Big] \end{aligned}$$

Exercise 22

Solve the given initial value problem.

$$\begin{aligned}
y'' + 2y' + 17y &= 0 \\
r^2 + 2r + 17 &= 0 \\
r &= \frac{-2 \pm \sqrt{4 - 4(17)}}{2} \\
&= -1 \pm 4i \quad \alpha = -1 \quad \beta = 4 \\
y &= \mathrm{e}^{-t}\Big[c_1 \cos(4t) + c_2 \sin(4t)\Big]
\end{aligned}$$

Given the initial values of $y(0) = 1$ and $y'(0) = -1$:

$$\begin{aligned}
y(t) &= \mathrm{e}^{-t}\Big[c_1 \cos(4t) + c_2 \sin(4t)\Big] \\
y(0) &= 1 = c_1 \\
y'(t) &= \mathrm{e}^{-t}\Big[-4c_1 \sin(4t) + 4c_2 \cos(4t)\Big] - \mathrm{e}^{-t}\Big[c_1 \cos(4t) + c_2 \sin(4t)\Big] \\
y'(0) &= -1 = 4c_2 - c_1 \\
c_2 &= 0 \\
y(t) &= \mathrm{e}^{-t} \cos(4t)
\end{aligned}$$

Exercise 23

Solve the given initial value problem.

$$\begin{aligned}
w'' - 4w' + 2w &= 0 \\
r^2 - 4r + 2 &= 0 \\
r &= \frac{4 \pm \sqrt{16 - 4(2)}}{2} \\
&= 2 \pm \sqrt{2} \\
w &= c_1 \mathrm{e}^{(2+\sqrt{2})t} + c_2 \mathrm{e}^{(2-\sqrt{2})t}
\end{aligned}$$

Given the initial values of $w(0) = 0$ and $w'(0) = 1$:

$$\begin{aligned}
w(t) &= c_1 \mathrm{e}^{(2+\sqrt{2})t} + c_2 \mathrm{e}^{(2-\sqrt{2})t} \\
w(0) &= 0 = c_1 + c_2 \\
w'(t) &= (2 + \sqrt{2})c_1 \mathrm{e}^{(2+\sqrt{2})t} + (2 - \sqrt{2})c_2 \mathrm{e}^{(2-\sqrt{2})t} \\
w'(0) &= 1 = (2 + \sqrt{2})c_1 + (2 - \sqrt{2})c_2 \\
c_1 &= \frac{\sqrt{2}}{4} \quad c_2 = -\frac{\sqrt{2}}{4} \\
w(t) &= \frac{\sqrt{2}}{4}\mathrm{e}^{(2+\sqrt{2})t} - \frac{\sqrt{2}}{4}\mathrm{e}^{(2-\sqrt{2})t} \\
&= \frac{\sqrt{2}}{4}\Big[\mathrm{e}^{(2+\sqrt{2})t} - \mathrm{e}^{(2-\sqrt{2})t}\Big]
\end{aligned}$$

Exercise 24

Solve the given initial value problem.

$$\begin{aligned} y'' + 9y &= 0 \\ r^2 + 9 &= 0 \\ r &= 0 \pm 3i \quad \alpha = 0 \quad \beta = 3 \\ y &= e^0 \left[c_1 \cos(3t) + c_2 \sin(3t) \right] \end{aligned}$$

Given the initial values of $y(0) = 1$ and $y'(0) = 1$:

$$\begin{aligned} y(t) &= c_1 \cos(3t) + c_2 \sin(3t) \\ y(0) &= 1 = c_1 \\ y'(t) &= -3c_1 \sin(3t) + 3c_2 \sin(3t) \\ y'(0) &= 1 = 3c_2 \\ c_2 &= \frac{1}{3} \\ y(t) &= \cos(3t) + \frac{\sin(3t)}{3} \end{aligned}$$

Exercise 26

Solve the given initial value problem.

$$\begin{aligned} y'' - 2y' + y &= 0 \\ r^2 - 2r + 1 &= 0 \\ (r-1)^2 &= 0 \\ r &= 1 \\ y &= c_1 e^t + c_2 t e^t \end{aligned}$$

Given the initial values of $y(0) = 1$ and $y'(0) = -2$:

$$\begin{aligned} y(t) &= c_1 e^t + c_2 t e^t \\ y(0) &= 1 = c_1 \\ y'(t) &= c_1 e^t + c_2 \left[e^t + t e^t \right] \\ y'(0) &= -2 = c_1 + c_2 \\ c_2 &= -3 \\ y(t) &= e^t - 3te^t \end{aligned}$$

Exercise 27

Solve the given initial value problem.

$$\begin{aligned} y''' - 4y'' + 7y' - 6y &= 0 \\ r^3 - 4r^2 + 7r - 6 &= 0 \\ (r-2)(r^2 - 2r + 3) &= 0 \\ r &= 2 \\ r &= \frac{2 \pm \sqrt{4 - 4(3)}}{2} \\ &= 1 \pm \sqrt{2}i \quad \alpha = 1 \quad \beta = \sqrt{2} \\ y &= c_1 e^{2t} + e^t \left[c_2 \cos(\sqrt{2}t) + c_3 \sin(\sqrt{2}t) \right] \end{aligned}$$

Given the initial values of $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 0$:

$$\begin{aligned} y(t) &= c_1 e^{2t} + e^t \left[c_2 \cos(\sqrt{2}t) + c_3 \sin(\sqrt{2}t) \right] \\ y(0) &= 1 = c_1 + c_2 \\ y'(t) &= 2c_1 e^{2t} + e^t \left[-\sqrt{2}c_2 \sin(\sqrt{2}t) + \sqrt{2}c_3 \cos(\sqrt{2}t) \right] + e^t \left[c_2 \cos(\sqrt{2}t) + c_3 \sin(\sqrt{2}t) \right] \\ y'(0) &= 0 = 2c_1 + \sqrt{2}c_3 + c_2 \\ y''(t) &= 4c_1 e^{2t} + e^t \left[-2c_2 \cos(\sqrt{2}t) - 2c_2 \sin(\sqrt{2}t) \right] + 2e^t \left[-\sqrt{2}c_2 \sin(\sqrt{2}t) + \sqrt{2}c_3 \cos(\sqrt{2}t) \right] + \\ &\quad e^t \left[c_2 \cos(\sqrt{2}t) + c_3 \sin(\sqrt{2}t) \right] \\ y''(0) &= 0 = 4c_1 - 2c_2 + 2\sqrt{2}c_3 + c_2 \\ c_1 &= 1 \quad c_2 = 0 \quad c_3 = -\sqrt{2} \\ y(t) &= e^{2t} - e^t \sqrt{2} \sin(\sqrt{2}t) \end{aligned}$$

Exercise 28

To see the effect of changing the parameter b in the initial value problem

$$y'' + by' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 0$$

solve the problem for $b = 5, 4, 2$ and sketch the solutions. $b = 5$:

$$\begin{aligned} y'' + 5y' + 4y &= 0 \\ r^2 + 5r + 4 &= (r+4)(r+1) = 0 \\ r &= -4 \quad r = -1 \\ y(t) &= c_1 e^{-4t} + c_2 e^{-t} \\ y(0) &= 1 = c_1 + c_2 \\ y'(t) &= -4c_1 e^{-4t} - c_2 e^{-t} \\ y'(0) &= 0 = -4c_1 - c_2 \\ c_1 &= -\frac{1}{3} \quad c_2 = \frac{4}{3} \\ y &= -\frac{1}{3} e^{-4t} + \frac{4}{3} e^{-t} \end{aligned}$$

$b = 4$:

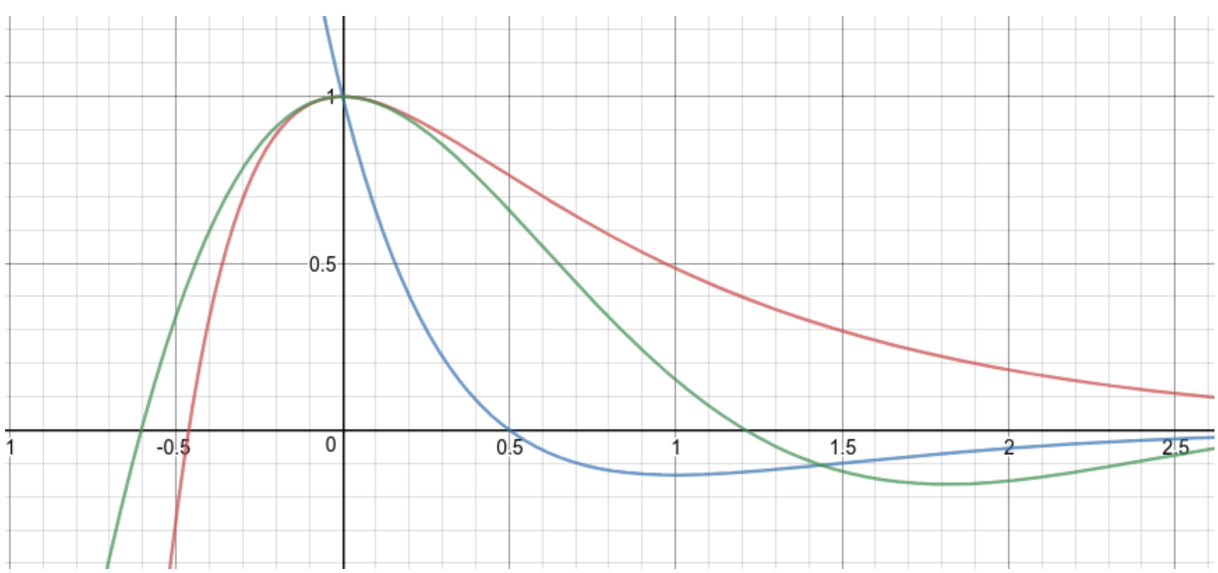
$$\begin{aligned}
y'' + 4y' + 4y &= 0 \\
r^2 + 4r + 4 &= (r+2)^2 = 0 \\
r &= -2 \\
y(t) &= c_1 \mathrm{e}^{-2t} + c_2 t\mathrm{e}^{-2t} \\
y(0) &= 1 = c_1 \\
y'(t) &= -2c_1 \mathrm{e}^{-2t} - 2c_2 t\mathrm{e}^{-2t} + c_2 \mathrm{e}^{2t} \\
y'(0) &= 0 = -2c_1 - c_2 \\
c_2 &= -2 \\
y(t) &= \mathrm{e}^{-2t} - 2t\mathrm{e}^{-2t}
\end{aligned}$$

$b = 2$:

$$\begin{aligned}
y'' + 2y' + 4y &= 0 \\
r^2 + 2r + 4 &= 0
\end{aligned}$$

$$\begin{aligned}
r &= \frac{-2 \pm \sqrt{4 - 4(4)}}{2} \\
&= -1 \pm \sqrt{3}i \quad \alpha = -1 \quad \beta = \sqrt{3} \\
y(t) &= \mathrm{e}^{-t}\Big[c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)\Big] \\
y(0) &= 1 = c_1 \\
y'(t) &= \mathrm{e}^{-t}\Big[-\sqrt{3}c_1 \sin(\sqrt{3}t) + \sqrt{3}c_2 \cos(\sqrt{3}t)\Big] + \mathrm{e}^{-t}\Big[c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)\Big]
\end{aligned}$$

$$\begin{aligned}
y'(0) &= 0 = -\sqrt{3}c_2 + c_1 \\
c_2 &= \frac{1}{\sqrt{3}} \\
y(t) &= \mathrm{e}^{-t}\left[\cos(\sqrt{3}t) + \frac{\sin(\sqrt{3}t)}{\sqrt{3}}\right]
\end{aligned}$$



Section 4.4

Exercise 1

Decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

$$y'' + 2y' - y = t^{-1}\mathrm{e}^{t}$$

$\frac{\mathrm{e}^t}{t}$ cannot be formed as $Ct^m\mathrm{e}^{rt}$, so the method of undetermined coefficients cannot be applied.

Exercise 4

Decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

$$x'' + 5x' - 3x = 3^t$$

3^t can be rewritten as $\mathrm{e}^{\ln(3)t}$, so the method of undetermined coefficients can be applied.

Exercise 5

Decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

$$y''(\theta) + 3y'(\theta) - y(\theta) = \sec(\theta)$$

$\sec\theta = \frac{1}{\cos\theta}$ cannot be expressed as a product of polynomials, expoentials, and $\sin, \cos$, so the method of undetermined coefficients cannot be applied.

Exercise 9

Find a particular solution to the differential equation.

$$\begin{aligned} y'' + 3y &= -9 \\ y_p &= A \\ y_p'' &= 0 \\ 0 + 3A &= -9 \\ y_p &= A = -3 \end{aligned}$$

Exercise 10

Find a particular solution to the differential equation.

$$\begin{aligned} y'' + 2y' - y &= 10 \\ y_p &= A \\ y_p' &= 0 \\ y_p'' &= 0 \\ 0 - 0 - A &= 10 \\ y_p &= A = -10 \end{aligned}$$

Exercise 12

Find a particular solution to the differential equation.

$$\begin{aligned}
2x' + x &= 3t^2 \\
x_p &= At^2 + Bt + C \\
x_p' &= 2At + B \\
2(2At + B) + At^2 + Bt + C &= 3t^2 \\
4At + 2B + At^2 + Bt + C &= 3t^2 \\
At^2 + (4A + B)t + (2B + C) &= 3t^2 \\
A &= 3 \quad 4A + B = 0 \quad 2B + C = 0 \\
A &= 3 \quad B = -12 \quad c = 24 \\
x_p &= 3t^2 - 12t + 24
\end{aligned}$$

Exercise 14

Find a particular solution to the differential equation.

$$\begin{aligned}
2z'' + z &= 9\mathrm{e}^{2t} \\
z_p &= A\mathrm{e}^{2t} \\
z_p' &= 2A\mathrm{e}^{2t} \\
z_p'' &= 4A\mathrm{e}^{2t} \\
2(4A\mathrm{e}^{2t}) + A\mathrm{e}^{2t} &= 9\mathrm{e}^{2t} \\
8A + A &= 9 \\
A &= 1 \\
z_p &= \mathrm{e}^{2t}
\end{aligned}$$

Exercise 18

Find a particular solution to the differential equation.

$$\begin{aligned}
y'' - 2y' + y &= 8\mathrm{e}^t \\
y_p = y_p' &= y_p'' = A\mathrm{e}^t \\
A\mathrm{e}^t - 2(A\mathrm{e}^t) + A\mathrm{e}^t &= 8\mathrm{e}^t \\
0 &= 8\mathrm{e}^t
\end{aligned}$$

This does not work since the auxiliary equation $r^2 - 2r + 1$ has the double root $r = 1$.

$$\begin{aligned}
y_p &= At^2\mathrm{e}^t \\
y_p' &= At^2\mathrm{e}^t + 2At\mathrm{e}^t = \mathrm{e}^t(At^2 + 2At) \\
y_p'' &= At^2\mathrm{e}^t + 2At\mathrm{e}^t + 2At\mathrm{e}^t + 2A\mathrm{e}^t \\
&= \mathrm{e}^t(At^2 + 4At + 2A) \\
\mathrm{e}^t(At^2 + 4At + 2A) - 2(At^2\mathrm{e}^t + 2At\mathrm{e}^t) + At^2\mathrm{e}^t &= 8\mathrm{e}^t \\
At^2 + 4At + 2A - 2At^2 - 4At + At^2 &= 8 \\
2A &= 8 \\
A &= 4 \\
y_p &= 4t^2\mathrm{e}^t
\end{aligned}$$

Exercise 23

Find a particular solution to the differential equation.

$$\begin{aligned}
y''(\theta) - 7y'(\theta) &= \theta^2 \\
r^2 - 7r &= 0 \\
r &= 0 \quad r = 7 \\
y''(\theta) - 7y'(\theta) &= 1\theta^2 \mathrm{e}^{0t} \\
y_p &= \theta(A\theta^2 + B\theta + C) \\
y_p' &= 3A\theta^2 + 2B\theta + C \\
y_p'' &= 6A\theta + 2B \\
6A\theta + 2B - 7(3A\theta^2 + 2B\theta + C) &= \theta^2 \\
-21A\theta^2 + (6A - 14B)\theta + (6A + 2B - 7C) &= \theta^2 \\
-21A &= 1 \quad 6A - 14B = 0 \quad 2B - 7C = 0 \\
A &= -\frac{1}{21} \quad B = -\frac{1}{49} \quad C = -\frac{2}{343} \\
y_p &= -\frac{\theta^3}{21} - \frac{\theta^2}{49} - \frac{2\theta}{343}
\end{aligned}$$

Exercise 28

Determine the form of a particular solution for the differential equation.

$$\begin{aligned}
y'' - 6y' + 9y &= 5t^6 \mathrm{e}^{3t} \\
C = 5 \quad m &= 6 \quad r = 3 \\
r^2 - 6r + 9 &= 0 \\
(r - 3)^3 &= 0 \\
r &= 3 \\
y_p(t) &= t^2(At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft + G)\mathrm{e}^{rt}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech