Differential Equations: Homework 5

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Section 3.5

Exercise 1

An RL circuit with a $5-\Omega$ resistor and a 0.05-H inductor carries a current of 1 A at $t=0$, at which time a voltage source $E(t)=5\cos(120t)V$ is added. Determine the subsequent inductor current and voltage.

$$\begin{aligned}
L\frac{\mathrm{d}i}{\mathrm{d}t}+Ri &= E(t)\\
\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{R}{L}i &= \frac{E(t)}{L}\\
\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{5}{0.05}i &= \frac{5\cos(120t)}{0.05}\\
\frac{\mathrm{d}i}{\mathrm{d}t}+100i &= 100\cos(120t)\\
\mu(t) &= \mathrm{e}^{\int 100\ \mathrm{d}t}=\mathrm{e}^{100t}\\
\mathrm{e}^{100t}\frac{\mathrm{d}i}{\mathrm{d}t}+\mathrm{e}^{100t}100i &= 100\cos(120t)\mathrm{e}^{100t}\\
\int\left(\mathrm{e}^{100t}\frac{\mathrm{d}i}{\mathrm{d}t}+\mathrm{e}^{100t}100i\right)\ \mathrm{d}t &= \int 100\cos(120t)\mathrm{e}^{100t}\ \mathrm{d}t\\
i\mathrm{e}^{100t}+c &= \int 100\cos(120t)\mathrm{e}^{100t}\ \mathrm{d}t\\
i &= 100\mathrm{e}^{-100t}\left(\int\cos(120t)\mathrm{e}^{100t}\ \mathrm{d}t+c\right)\\
&= 100\mathrm{e}^{-100t}\left[\frac{\mathrm{e}^{100t}(100\cos(120t)+120\sin(120t))}{100^2+120^2}+c\right]\\
&= \frac{100\cos(120t)+120\sin(120t)}{244}+100c\mathrm{e}^{-100t}
\end{aligned}$$

Using our initial value of $i(0)=1$:

$$\begin{aligned}
i(t) &= \frac{100\cos(120t)+120\sin(120t)}{244}+k\mathrm{e}^{-100t}\\
1=i(0) &= \frac{100\cos(0)+120\sin(0)}{244}+k\mathrm{e}^{0}\\
&= \frac{100}{244}+k\\
k &= \frac{144}{244}\\
i(t) &= \frac{100\cos(120t)+120\sin(120t)}{244}+\frac{144}{244}\mathrm{e}^{-100t}\\
i(t) &= \frac{100\cos(120t)+120\sin(120t)+144\mathrm{e}^{-100t}}{244}
\end{aligned}$$

To find the voltage drop:

$$L\frac{\mathrm{d}i}{\mathrm{d}t}=0.5\left(\frac{-12000\cos(120t)+12000\cos(120t)-14400\mathrm{e}^{-100t}}{244}\right)$$

Exercise 7

An industrial electromagnet can be modeled as an RL circuit, while it is being energized with a voltage source. If the inductance is 10H and the wire windings contain 3Ω of resistance, how long does it take a constant applied voltage to energize the electromagnet to within 90% of its final value (that is, the current equals 90% of its asymptotic value)?

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}i = \frac{E}{L}$$
$$\mu(t) = \mathrm{e}^{\int \frac{R}{L}\,\mathrm{d}t} = \mathrm{e}^{\frac{Rt}{L}}$$
$$\mathrm{e}^{\frac{Rt}{L}}\frac{\mathrm{d}i}{\mathrm{d}t} + \mathrm{e}^{\frac{Rt}{L}}\frac{R}{L}i = \mathrm{e}^{\frac{Rt}{L}}\frac{E}{L}$$
$$ie^{\frac{R}{L}t} + c = \int \mathrm{e}^{\frac{Rt}{L}}\frac{E}{L}\,\mathrm{d}t$$
$$i = \frac{E}{R} + c\mathrm{e}^{-\frac{Rt}{L}}$$

$$\begin{aligned}
0 &= \frac{E}{R} + c\mathrm{e}^{0} \\
c &= -\frac{E}{R} \\
i &= \frac{E}{R}(1 - \mathrm{e}^{-\frac{Rt}{L}}) \\
\lim_{t\to\infty} i &= \frac{E}{R} \\
0.9\frac{E}{R} &= i = \frac{E}{R}(1 - \mathrm{e}^{-\frac{Rt}{L}}) \\
0.9 &= 1 - \mathrm{e}^{-\frac{Rt}{L}} \\
\mathrm{e}^{-\frac{Rt}{L}} &= 0.1 \\
-\frac{Rt}{L} &= \ln(0.1) \\
t &= \frac{-L\ln(0.1)}{R} \\
&= \frac{-10\ln(0.1)}{3} \approx 7.67
\end{aligned}$$

Section 4.2

Exercise 1

Find a general solution to the given differential equation.

$$\begin{aligned} 2y'' + 7y' - 4y &= 0 \\ y(t) &= \mathrm{e}^{rt} \\ y'(t) &= r\mathrm{e}^{rt} \\ y''(t) &= r^2\mathrm{e}^{rt} \\ 2r^2\mathrm{e}^{rt} + 7r\mathrm{e}^{rt} - 4\mathrm{e}^{rt} &= 0 \\ 2r^2 + 7r - 4 &= 0 \\ (2r-1)(r+4) &= 0 \\ r &= \frac{1}{2} \quad r = -4 \\ y &= \mathrm{e}^{\frac{1}{2}t} \quad y = \mathrm{e}^{-4t} \\ y &= c_1\mathrm{e}^{\frac{1}{2}t} + c_2\mathrm{e}^{-4t} \end{aligned}$$

Exercise 2

Find a general solution to the given differential equation.

$$\begin{aligned} y'' + 6y' + 9y &= 0 \\ r^2 + 6r + 9 &= (r+3)(r+3) = 0 \\ r &= -3 \\ y &= c_1\mathrm{e}^{-3t} + c_2t\mathrm{e}^{-3t} \end{aligned}$$

Exercise 3

Find a general solution to the given differential equation.

$$\begin{aligned} y'' + 5y' + 6y &= 0 \\ r^2 + 5r + 6 &= (r+3)(r+2) = 0 \\ r &= -3 \quad r = -2 \\ y &= \mathrm{e}^{-3t} \quad y = \mathrm{e}^{-2t} \\ y &= c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-2t} \end{aligned}$$

Exercise 4

Find a general solution to the given differential equation.

$$\begin{aligned} y'' - y' - 2y &= 0 \\ r^2 - r - 2 &= (r-2)(r+1) = 0 \\ r &= 2 \quad r = -1 \\ y &= \mathrm{e}^{2t} \quad y = \mathrm{e}^{-t} \\ y &= c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{-t} \end{aligned}$$

Exercise 7

Find a general solution to the given differential equation.

$$\begin{aligned}6y'' + y' - 2y &= 0\\ 6r^2 + r - 2 = (2r-1)(3r+2) &= 0\\ r = \frac{1}{2} \quad r &= -\frac{2}{3}\\ y = \mathrm{e}^{\frac{1}{2}t} \quad y &= \mathrm{e}^{-\frac{2}{3}t}\\ y &= c_1\mathrm{e}^{\frac{1}{2}t} + c_2\mathrm{e}^{-\frac{2}{3}t}\end{aligned}$$

Exercise 8

Find a general solution to the given differential equation.

$$\begin{aligned}z'' + z' - z &= 0\\ r^2 + r - 1 &= 0\\ r &= \frac{-1 \pm \sqrt{5}}{2}\\ z = \mathrm{e}^{\frac{-1+\sqrt{5}}{2}t} \quad z &= \mathrm{e}^{\frac{-1-\sqrt{5}}{2}t}\\ z &= c_1\mathrm{e}^{\frac{-1+\sqrt{5}}{2}t} + c_2\mathrm{e}^{\frac{-1-\sqrt{5}}{2}t}\end{aligned}$$

Exercise 11

Find a general solution to the given differential equation.

$$\begin{aligned}4w'' + 20w' + 25w &= 0\\ 4r^2 + 20r + 25 = (2r+5)^2 &= 0\\ r &= -\frac{5}{2}\\ w &= c_1\mathrm{e}^{-\frac{5}{2}t} + c_2t\mathrm{e}^{-\frac{5}{2}t}\end{aligned}$$

Exercise 13

Solve the given initial value problem.

$$\begin{aligned}y'' + 2y' - 8y &= 0\\ r^2 + 2r - 8 = (r+4)(r-2) &= 0\\ r = -4 \quad r &= 2\\ y = \mathrm{e}^{-4t} \quad y &= \mathrm{e}^{2t}\\ y &= c_1\mathrm{e}^{-4t} + c_2\mathrm{e}^{2t}\end{aligned}$$

Using the initial values of $y(0) = 3$ and $y'(0) = -12$:

$$\begin{aligned}y &= c_1\mathrm{e}^{-4t} + c_2\mathrm{e}^{2t}\\ 3 &= c_1 + c_2\\ y' &= -4c_1\mathrm{e}^{-4t} + 2c_2\mathrm{e}^{2t}\\ -12 &= -4c_1 + 2c_2\\ c_1 = 3 \quad c_2 &= 0\\ y &= 3\mathrm{e}^{-4t}\end{aligned}$$

Exercise 14

Solve the given initial value problem.

$$\begin{aligned} y'' + y' &= 0 \\ r^2 + r &= r(r+1) = 0 \\ r &= 0 \quad r = -1 \\ y &= \mathrm{e}^{0t} = 1 \quad y = \mathrm{e}^{-t} \\ y &= c_1 + c_2\mathrm{e}^{-t} \end{aligned}$$

Using the initial values of $y(0) = 2$ and $y'(0) = 1$:

$$\begin{aligned} y &= c_1 + c_2\mathrm{e}^{-t} \\ 2 &= c_1 + c_2 \\ y' &= -c_2\mathrm{e}^{-t} \\ 1 &= -c_2 \\ c_1 &= 3 \quad c_2 = -1 \\ y &= -\mathrm{e}^{-3} + 3 \end{aligned}$$

Exercise 15

Solve the given initial value problem.

$$\begin{aligned} y'' - 4y' + 3y &= 0 \\ r^2 - 4r + 3 &= (r-3)(r-1) = 0 \\ r &= 3 \quad r = 1 \\ y &= \mathrm{e}^{3t} \quad y = \mathrm{e}^{t} \\ y &= c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{t} \end{aligned}$$

Using the initial value of $y(0) = 1$ and $y'(0) = \frac{1}{3}$:

$$\begin{aligned} y &= c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{t} \\ 1 &= c_2 + c_2 \\ y' &= 3c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{t} \\ \frac{1}{3} &= 3c_1 + c_2 \\ c_1 &= -\frac{1}{3} \quad c_2 = \frac{4}{3} \\ y &= -\frac{1}{3}\mathrm{e}^{3t} + \frac{4}{3}\mathrm{e}^{t} \end{aligned}$$

Exercise 17

Solve the given initial value problem.

$$\begin{aligned} y'' - 6y' + 9y &= 0 \\ r^2 - 6r + 9 &= (r-3)^2 = 0 \\ r &= 3 \\ y &= \mathrm{e}^{3t} \\ y &= c_1\mathrm{e}^{3t} + c_2t\mathrm{e}^{3t} \end{aligned}$$

Using the initial values of $y(0) = 2$ and $y'(0) = \frac{25}{3}$:

$$\begin{aligned} y &= c_1 e^{3t} + c_2 t e^{3t} \\ 2 &= c_1 \\ y' &= 3c_1 e^{3t} + c_2 e^{3t}(3t+1) \\ \frac{25}{3} &= 3c_1 + c_2 \\ c_1 &= 2 \quad c_2 = \frac{7}{3} \\ y &= 2e^{3t} + \frac{7}{3} t e^{3t} \end{aligned}$$

Exercise 18

Solve the given initial value problem.

$$\begin{aligned} z'' - 2z' - 2z &= 0 \\ r^2 - 2r - 2 &= 0 \end{aligned}$$

$$\begin{aligned} r &= \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3} \\ z &= c_1 e^{t(1+\sqrt{3})} + c_2 e^{t(1-\sqrt{3})} \end{aligned}$$

Using the initial values of $z(0) = 0$ and $z'(0) = 3$:

$$\begin{aligned} z &= c_1 e^{t(1+\sqrt{3})} + c_2 e^{t(1-\sqrt{3})} \\ 0 &= c_1 + c_2 \\ z' &= (1+\sqrt{3}) c_1 e^{t(1+\sqrt{3})} + (1-\sqrt{3}) e^{t(1-\sqrt{3})} \\ 3 &= (1+\sqrt{3}) c_1 + (1-\sqrt{3}) c_2 \\ c_1 &= \frac{\sqrt{3}}{2} \quad c_2 = -\frac{\sqrt{3}}{2} \\ z &= \frac{\sqrt{3}}{2} e^{t(1+\sqrt{3})} - \frac{\sqrt{3}}{2} e^{t(1-\sqrt{3})} \end{aligned}$$

Exercise 19

Solve the given initial value problem.

$$\begin{aligned} y'' + 2y' + y &= 0 \\ r^2 + 2r + 1 &= (r+1)^2 = 0 \\ r &= -1 \\ y &= c_1 e^{-t} + c_2 t e^{-t} \end{aligned}$$

Using the initial values of $y(0) = 1$ and $y'(0) = -3$:

$$\begin{aligned} y &= c_1 e^{-t} + c_2 t e^{-t} \\ 1 &= c_1 \\ y' &= -c_1 e^{-t} - c_2 e^{-t}(t-1) \\ -3 &= -c_1 + c_2 \\ c_1 &= 1 \quad c_2 = -2 \\ y &= e^{-t} - 2t e^{-t} \end{aligned}$$

Exercise 29

Use Definition 1 to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$:

$$y_1(t) = t\mathrm{e}^{2t} \quad y_2(t) = \mathrm{e}^{2t}$$

y_1 and y_2 are linearly independent because they are not *constant multiples* of one another.

Exercise 30

Use Definition 1 to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$:

$$y_1(t) = t^2 \cos(\ln(t)) \quad y_2(t) = t^2 \sin(\ln(t))$$

y_1 and y_2 are linearly independent on the interval $(0, 1)$ because they are not *constant multiples* of one another. They are only constant multiples when $t = 0$.

Exercise 32

Use Definition 1 to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$:

$$y_1(t) = 0 \quad y_2(t) = \mathrm{e}^{t}$$

Since $y_1(t) = 0$, it's only possible multiple is 0, therefore the two functions are linearly independent.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech