

Differential Equations: Homework 5

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Section 3.5

Exercise 1

An RL circuit with a $5 - \Omega$ resistor and a 0.05-H inductor carries a current of 1 A at $t = 0$, at which time a voltage source $E(t) = 5 \cos(120t)\text{V}$ is added. Determine the subsequent inductor current and voltage.

$$\begin{aligned}
 L \frac{di}{dt} + Ri &= E(t) \\
 \frac{di}{dt} + \frac{R}{L}i &= \frac{E(t)}{L} \\
 \frac{di}{dt} + \frac{5}{0.05}i &= \frac{5 \cos(120t)}{0.05} \\
 \frac{di}{dt} + 100i &= 100 \cos(120t) \\
 \mu(t) &= e^{\int 100 dt} = e^{100t} \\
 e^{100t} \frac{di}{dt} + e^{100t} 100i &= 100 \cos(120t)e^{100t} \\
 \int \left(e^{100t} \frac{di}{dt} + e^{100t} 100i \right) dt &= \int 100 \cos(120t)e^{100t} dt \\
 ie^{100t} + c &= \int 100 \cos(120t)e^{100t} dt \\
 i &= 100e^{-100t} \left(\int \cos(120t)e^{100t} dt + c \right) \\
 &= 100e^{-100t} \left[\frac{e^{100t}(100 \cos(120t) + 120 \sin(120t))}{100^2 + 120^2} + c \right] \\
 &= \frac{100 \cos(120t) + 120 \sin(120t)}{244} + 100ce^{-100t}
 \end{aligned}$$

Using our initial value of $i(0) = 1$:

$$\begin{aligned}
 i(t) &= \frac{100 \cos(120t) + 120 \sin(120t)}{244} + ke^{-100t} \\
 1 &= i(0) = \frac{100 \cos(0) + 120 \sin(0)}{244} + ke^0 \\
 &= \frac{100}{244} + k \\
 k &= \frac{144}{244} \\
 i(t) &= \frac{100 \cos(120t) + 120 \sin(120t)}{244} + \frac{144}{244}e^{-100t} \\
 i(t) &= \frac{100 \cos(120t) + 120 \sin(120t) + 144e^{-100t}}{244}
 \end{aligned}$$

To find the voltage drop:

$$L \frac{di}{dt} = 0.5 \left(\frac{-12000 \cos(120t) + 12000 \cos(120t) - 14400e^{-100t}}{244} \right)$$

Exercise 7

An industrial electromagnet can be modeled as an RL circuit, while it is being energized with a voltage source. If the inductance is 10H and the wire windings contain 3Ω of resistance, how long does it take a constant applied voltage to energize the electromagnet to within 90% of its final value (that is, the current equals 90% of its asymptotic value)?

$$\begin{aligned}\frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \\ \mu(t) &= e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}} \\ e^{\frac{Rt}{L}} \frac{di}{dt} + e^{\frac{Rt}{L}} \frac{R}{L}i &= e^{\frac{Rt}{L}} \frac{E}{L} \\ ie^{\frac{Rt}{L}} + c &= \int e^{\frac{Rt}{L}} \frac{E}{L} dt \\ i &= \frac{E}{R} + ce^{-\frac{Rt}{L}}\end{aligned}$$

$$\begin{aligned}0 &= \frac{E}{R} + ce^0 \\ c &= -\frac{E}{R} \\ i &= \frac{E}{R}(1 - e^{-\frac{Rt}{L}}) \\ \lim_{t \rightarrow \infty} i &= \frac{E}{R} \\ 0.9 \frac{E}{R} &= i = \frac{E}{R}(1 - e^{-\frac{Rt}{L}}) \\ 0.9 &= 1 - e^{-\frac{Rt}{L}} \\ e^{-\frac{Rt}{L}} &= 0.1 \\ -\frac{Rt}{L} &= \ln(0.1) \\ t &= \frac{-L \ln(0.1)}{R} \\ &= \frac{-10 \ln(0.1)}{3} \approx 7.67\end{aligned}$$

Section 4.2

Exercise 1

Find a general solution to the given differential equation.

$$\begin{aligned}2y'' + 7y' - 4y &= 0 \\y(t) &= e^{rt} \\y'(t) &= re^{rt} \\y''(t) &= r^2e^{rt} \\2r^2e^{rt} + 7re^{rt} - 4e^{rt} &= 0 \\2r^2 + 7r - 4 &= 0 \\(2r - 1)(r + 4) &= 0 \\r = \frac{1}{2} \quad r = -4 \\y = e^{\frac{1}{2}t} \quad y = e^{-4t} \\y &= c_1e^{\frac{1}{2}t} + c_2e^{-4t}\end{aligned}$$

Exercise 2

Find a general solution to the given differential equation.

$$\begin{aligned}y'' + 6y' + 9y &= 0 \\r^2 + 6r + 9 &= (r + 3)(r + 3) = 0 \\r &= -3 \\y &= c_1e^{-3t} + c_2te^{-3t}\end{aligned}$$

Exercise 3

Find a general solution to the given differential equation.

$$\begin{aligned}y'' + 5y' + 6y &= 0 \\r^2 + 5r + 6 &= (r + 3)(r + 2) = 0 \\r = -3 \quad r = -2 \\y = e^{-3t} \quad y = e^{-2t} \\y &= c_1e^{-3t} + c_2e^{-2t}\end{aligned}$$

Exercise 4

Find a general solution to the given differential equation.

$$\begin{aligned}y'' - y' - 2y &= 0 \\r^2 - r - 2 &= (r - 2)(r + 1) = 0 \\r = 2 \quad r = -1 \\y = e^{2t} \quad y = e^{-t} \\y &= c_1e^{2t} + c_2e^{-t}\end{aligned}$$

Exercise 7

Find a general solution to the given differential equation.

$$\begin{aligned}6y'' + y' - 2y &= 0 \\6r^2 + r - 2 &= (2r - 1)(3r + 2) = 0 \\r &= \frac{1}{2} \quad r = -\frac{2}{3} \\y &= e^{\frac{1}{2}t} \quad y = e^{-\frac{2}{3}t} \\y &= c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{2}{3}t}\end{aligned}$$

Exercise 8

Find a general solution to the given differential equation.

$$\begin{aligned}z'' + z' - z &= 0 \\r^2 + r - 1 &= 0 \\r &= \frac{-1 \pm \sqrt{5}}{2} \\z &= e^{\frac{-1+\sqrt{5}}{2}t} \quad z = e^{\frac{-1-\sqrt{5}}{2}t} \\z &= c_1 e^{\frac{-1+\sqrt{5}}{2}t} + c_2 e^{\frac{-1-\sqrt{5}}{2}t}\end{aligned}$$

Exercise 11

Find a general solution to the given differential equation.

$$\begin{aligned}4w'' + 20w' + 25w &= 0 \\4r^2 + 20r + 25 &= (2r + 5)^2 = 0 \\r &= -\frac{5}{2} \\w &= c_1 e^{-\frac{5}{2}t} + c_2 t e^{-\frac{5}{2}t}\end{aligned}$$

Exercise 13

Solve the given initial value problem.

$$\begin{aligned}y'' + 2y' - 8y &= 0 \\r^2 + 2r - 8 &= (r + 4)(r - 2) = 0 \\r &= -4 \quad r = 2 \\y &= e^{-4t} \quad y = e^{2t} \\y &= c_1 e^{-4t} + c_2 e^{2t}\end{aligned}$$

Using the initial values of $y(0) = 3$ and $y'(0) = -12$:

$$\begin{aligned}y &= c_1 e^{-4t} + c_2 e^{2t} \\3 &= c_1 + c_2 \\y' &= -4c_1 e^{-4t} + 2c_2 e^{2t} \\-12 &= -4c_1 + 2c_2 \\c_1 &= 3 \quad c_2 = 0 \\y &= 3e^{-4t}\end{aligned}$$

Exercise 14

Solve the given initial value problem.

$$\begin{aligned}
 y'' + y' &= 0 \\
 r^2 + r &= r(r + 1) = 0 \\
 r &= 0 \quad r = -1 \\
 y &= e^{0t} = 1 \quad y = e^{-t} \\
 y &= c_1 + c_2e^{-t}
 \end{aligned}$$

Using the initial values of $y(0) = 2$ and $y'(0) = 1$:

$$\begin{aligned}
 y &= c_1 + c_2e^{-t} \\
 2 &= c_1 + c_2 \\
 y' &= -c_2e^{-t} \\
 1 &= -c_2 \\
 c_1 &= 3 \quad c_2 = -1 \\
 y &= -e^{-3} + 3
 \end{aligned}$$

Exercise 15

Solve the given initial value problem.

$$\begin{aligned}
 y'' - 4y' + 3y &= 0 \\
 r^2 - 4r + 3 &= (r - 3)(r - 1) = 0 \\
 r &= 3 \quad r = 1 \\
 y &= e^{3t} \quad y = e^t \\
 y &= c_1e^{3t} + c_2e^t
 \end{aligned}$$

Using the initial value of $y(0) = 1$ and $y'(0) = \frac{1}{3}$:

$$\begin{aligned}
 y &= c_1e^{3t} + c_2e^t \\
 1 &= c_1 + c_2 \\
 y' &= 3c_1e^{3t} + c_2e^t \\
 \frac{1}{3} &= 3c_1 + c_2 \\
 c_1 &= -\frac{1}{3} \quad c_2 = \frac{4}{3} \\
 y &= -\frac{1}{3}e^{3t} + \frac{4}{3}e^t
 \end{aligned}$$

Exercise 17

Solve the given initial value problem.

$$\begin{aligned}
 y'' - 6y' + 9y &= 0 \\
 r^2 - 6r + 9 &= (r - 3)^2 = 0 \\
 r &= 3 \\
 y &= e^{3t} \\
 y &= c_1e^{3t} + c_2te^{3t}
 \end{aligned}$$

Using the initial values of $y(0) = 2$ and $y'(0) = \frac{25}{3}$:

$$\begin{aligned}y &= c_1 e^{3t} + c_2 t e^{3t} \\2 &= c_1 \\y' &= 3c_1 e^{3t} + c_2 e^{3t}(3t + 1) \\ \frac{25}{3} &= 3c_1 + c_2 \\c_1 &= 2 \quad c_2 = \frac{7}{3} \\y &= 2e^{3t} + \frac{7}{3}te^{3t}\end{aligned}$$

Exercise 18

Solve the given initial value problem.

$$\begin{aligned}z'' - 2z' - 2z &= 0 \\r^2 - 2r - 2 &= 0 \\r &= \frac{2 \pm \sqrt{4 + 8}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3} \\z &= c_1 e^{t(1+\sqrt{3})} + c_2 e^{t(1-\sqrt{3})}\end{aligned}$$

Using the initial values of $z(0) = 0$ and $z'(0) = 3$:

$$\begin{aligned}z &= c_1 e^{t(1+\sqrt{3})} + c_2 e^{t(1-\sqrt{3})} \\0 &= c_1 + c_2 \\z' &= (1 + \sqrt{3})c_1 e^{t(1+\sqrt{3})} + (1 - \sqrt{3})c_2 e^{t(1-\sqrt{3})} \\3 &= (1 + \sqrt{3})c_1 + (1 - \sqrt{3})c_2 \\c_1 &= \frac{\sqrt{3}}{2} \quad c_2 = -\frac{\sqrt{3}}{2} \\z &= \frac{\sqrt{3}}{2} e^{t(1+\sqrt{3})} - \frac{\sqrt{3}}{2} e^{t(1-\sqrt{3})}\end{aligned}$$

Exercise 19

Solve the given initial value problem.

$$\begin{aligned}y'' + 2y' + y &= 0 \\r^2 + 2r + 1 &= (r + 1)^2 = 0 \\r &= -1 \\y &= c_1 e^{-t} + c_2 t e^{-t}\end{aligned}$$

Using the initial values of $y(0) = 1$ and $y'(0) = -3$:

$$\begin{aligned}y &= c_1 e^{-t} + c_2 t e^{-t} \\1 &= c_1 \\y' &= -c_1 e^{-t} - c_2 e^{-t}(t - 1) \\-3 &= -c_1 + c_2 \\c_1 &= 1 \quad c_2 = -2 \\y &= e^{-t} - 2te^{-t}\end{aligned}$$

Exercise 29

Use Definition 1 to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$:

$$y_1(t) = te^{2t} \quad y_2(t) = e^{2t}$$

y_1 and y_2 are linearly independent because they are not *constant multiples* of one another.

Exercise 30

Use Definition 1 to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$:

$$y_1(t) = t^2 \cos(\ln(t)) \quad y_2(t) = t^2 \sin(\ln(t))$$

y_1 and y_2 are linearly independent on the interval $(0, 1)$ because they are not *constant multiples* of one another. They are only constant multiples when $t = 0$.

Exercise 32

Use Definition 1 to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$:

$$y_1(t) = 0 \quad y_2(t) = e^t$$

Since $y_1(t) = 0$, it's only possible multiple is 0, therefore the two functions are linearly independent.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech