Differential Equations: Homework 3

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January 2018 - May 2018

Section 2.3

Exercise 1

Determine whether the given equation is separable, linear, neither, or both.

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + \sin(x) - y = 0$$

Not separable and linear.

Exercise 3

Determine whether the given equation is separable, linear, neither, or both.

$$(t^2 + 1)\frac{\mathrm{d}y}{\mathrm{d}t} = yt - y$$

Separable and non-linear.

Exercise 5

Determine whether the given equation is separable, linear, neither, or both.

$$x\frac{\mathrm{d}x}{\mathrm{d}t} + t^2 x = \sin(t)$$

Not separable and non-linear.

Exercise 7

Obtain the general solution to the equation:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} - y - \mathrm{e}^{3x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} - y &= \mathrm{e}^{3x} \\
\mu(x) &= \mathrm{e}^{\int -1 \,\mathrm{d}x} = \mathrm{e}^{-x} \\
\mathrm{e}^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} - y\mathrm{e}^{-x} &= \mathrm{e}^{3x}\mathrm{e}^{-x} \\
\int \left(\mathrm{e}^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} - y\mathrm{e}^{-x}\right) \mathrm{d}x &= \int \mathrm{e}^{3x}\mathrm{e}^{-x} \,\mathrm{d}x \\
y\mathrm{e}^{-x} &= \frac{1}{2}\mathrm{e}^{2x} + c
\end{aligned}$$

Exercise 8

Obtain the general solution to the equation:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y}{x} + 2x + 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y &= 2x + 1 \\
\mu(x) &= \mathrm{e}^{\int \frac{-1}{x}\ \mathrm{d}x} = \mathrm{e}^{-\ln(x)} = \frac{1}{x} \\
\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - y &= 2 + \frac{1}{x} \\
\int \left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x^2}y\right)\ \mathrm{d}x &= \int 2 + \frac{1}{x}\ \mathrm{d}x \\
\frac{y}{x} &= 2x - \ln(x) + c \\
y &= 2x^2 - x\ln(x) + cx
\end{aligned}$$

Exercise 9

Obtain the general solution to the equation:

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} + r\tan\theta &= \sec\theta \\
\mu(\theta) &= \mathrm{e}^{\int \tan\theta\ \mathrm{d}\theta} = \mathrm{e}^{\ln(\sec\theta)} = \sec\theta \\
\sec(x)\frac{\mathrm{d}r}{\mathrm{d}\theta} + r\tan\theta\sec\theta &= \sec^2\theta \\
\int \left(\sec(x)\frac{\mathrm{d}r}{\mathrm{d}\theta} + r\tan\theta\sec\theta\right)\ \mathrm{d}\theta &= \int \sec^2\theta\ \mathrm{d}\theta \\
r\sec\theta &= \tan\theta + c \\
r &= \sin\theta + c\cos\theta
\end{aligned}$$

Exercise 10

Obtain the general solution to the equation:

$$\begin{aligned}
x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= x^{-3} \\
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y &= x^{-4} \\
\mu(x) &= \mathrm{e}^{\int \frac{2}{x}\ \mathrm{d}x} = \mathrm{e}^{2\ln(x)} = x^2 \\
x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy &= x^{-2} \\
\int \left(x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy\right)\ \mathrm{d}x &= \int x^{-2}\ \mathrm{d}x \\
yx^2 &= -\frac{1}{x} + c \\
y &= -\frac{1}{x^3} + \frac{c}{x^2}
\end{aligned}$$

Exercise 11

Obtain the general solution to the equation:

$$
\begin{aligned}
(t+y+1)\ \mathrm{d}t - \ \mathrm{d}y &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= t+y+1 \\
\frac{\mathrm{d}y}{\mathrm{d}t} - y &= t+1 \\
\mu(t) &= \mathrm{e}^{\int -1\ \mathrm{d}t} = \mathrm{e}^{-t} \\
\mathrm{e}^{-t}\frac{\mathrm{d}y}{\mathrm{d}t} - y\mathrm{e}^{-t} &= t\mathrm{e}^{-t} + \mathrm{e}^{-t} \\
\int \left(\mathrm{e}^{-t}\frac{\mathrm{d}y}{\mathrm{d}t} - y\mathrm{e}^{-t}\right) &= \int t\mathrm{e}^{-t} + \mathrm{e}^{-t}\ \mathrm{d}t \\
y\mathrm{e}^{-t} &= -e^{-t}(t+1) - \mathrm{e}^{-t} + c \\
y &= -t-1-1+c\mathrm{e}^{t} \\
&= -t-2+c\mathrm{e}^{t}
\end{aligned}
$$

Exercise 15

Obtain the general solution to the equation:

$$
\begin{aligned}
(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + xy - x &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y &= \frac{x}{x^2+1} \\
\mu(x) &= \mathrm{e}^{\int \frac{x}{x^2+1}\ \mathrm{d}x} = \mathrm{e}^{\frac{1}{2}\ln(x^2+1)} = \sqrt{x^2+1} \\
\sqrt{x^2+1}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{\sqrt{x^2+1}}y &= \frac{x}{\sqrt{x^2+1}} \\
\int \left(\sqrt{x^2+1}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{\sqrt{x^2+1}}y\right)\ \mathrm{d}x &= \int \frac{x}{\sqrt{x^2+1}}\ \mathrm{d}x \\
y\sqrt{x^2+1} &= \sqrt{x^2+1} + c \\
y &= 1 + \frac{c}{\sqrt{x^2+1}}
\end{aligned}
$$

Exercise 17

Solve the initial value problem.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} &= x\mathrm{e}^{x} \quad y(1) = \mathrm{e} - 1 \\
\mu(x) &= \mathrm{e}^{\int -\frac{1}{x}\ \mathrm{d}x} = \mathrm{e}^{-\ln(x)} = \frac{1}{x} \\
\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x^2}y &= \mathrm{e}^{x} \\
\int \left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x^2}y\right)\ \mathrm{d}x &= \int \mathrm{e}^{x}\ \mathrm{d}x \\
\frac{y}{x} &= \mathrm{e}^{x} + c \\
y &= x\mathrm{e}^{x} + cx
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{e} - 1 &= 1\mathrm{e}^1 + c1 \\
c &= -1 \\
y &= x\mathrm{e}^x - x
\end{aligned}
$$

Exercise 18

Solve the initial value problem.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - \mathrm{e}^{-x} = 0 \quad y(0) &= \frac{4}{3} \\
\frac{\mathrm{d}y}{\mathrm{d}x} + 4y &= \mathrm{e}^{-x} \\
\mu(x) &= \mathrm{e}^{\int 4 \ \mathrm{d}x} = \mathrm{e}^{4x} \\
\mathrm{e}^{4x}\frac{\mathrm{d}y}{\mathrm{d}x} + 4y\mathrm{e}^{4x} &= \mathrm{e}^{3x} \\
\int \left(\mathrm{e}^{4x}\frac{\mathrm{d}y}{\mathrm{d}x} + 4y\mathrm{e}^{4x}\right) \mathrm{d}x &= \int \mathrm{e}^{3x} \ \mathrm{d}x \\
y\mathrm{e}^{4x} &= \frac{1}{3}\mathrm{e}^{3x} + c \\
y &= \frac{1}{3\mathrm{e}^x} + \frac{c}{\mathrm{e}^{4x}} \\
\frac{4}{3} &= \frac{1}{3\mathrm{e}^0} + \frac{c}{\mathrm{e}^0} \\
c &= 1 \\
y &= \frac{1}{3\mathrm{e}^x} + \frac{1}{\mathrm{e}^{4x}}
\end{aligned}
$$

Exercise 20

Solve the initial value problem.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3y}{x} + 2 = 3x \quad y(1) &= 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3}{x}y &= 3x - 2 \\
\mu(x) &= \mathrm{e}^{\int \frac{3}{x} \ \mathrm{d}x} = \mathrm{e}^{3\ln(x)} = x^3 \\
x^3\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y &= 3x^4 - 2x^3 \\
\int \left(x^3\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y\right) \mathrm{d}x &= \int 3x^4 - 2x^3 \ \mathrm{d}x \\
yx^3 &= \frac{3}{5}x^5 - \frac{2}{4}x^4 + c \\
y &= \frac{3}{5}x^2 - \frac{1}{2}x^2 + \frac{c}{x^3} \\
1 &= \frac{3}{5} - \frac{1}{2} + c \quad c = \frac{9}{10} \\
y &= \frac{3}{5}x^2 - \frac{1}{2}x^2 + \frac{9}{10x^3}
\end{aligned}
$$

Section 2.6

Exercise 21

Use the method discussed under “Bernoulli Equations” to solve:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} &= x^2y^2 \\
v &= y^{-1} \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y^2}\frac{y}{x} &= \frac{1}{y^2}x^2y^2 \\
\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{xy} &= x^2 \\
-\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{v}{x} &= x^2 \\
\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v}{x} &= -x^2 \\
\mu(x) &= \mathrm{e}^{\int -\frac{1}{x}\,\mathrm{d}x} = \mathrm{e}^{-\ln(x)} = \frac{1}{x} \\
\frac{1}{x}\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v}{x^2} &= -x^2 \\
\int \left(\frac{1}{x}\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v}{x^2}\right)\mathrm{d}x &= \int -x^2\,\mathrm{d}x \\
\frac{v}{x} &= -\frac{x^3}{3} + c \\
\frac{1}{xy} &= -\frac{x^3}{3} + c
\end{aligned}$$

Exercise 22

Use the method discussed under “Bernoulli Equations” to solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y = \mathrm{e}^{2x}y^3$$

$$P(x) = -1 \quad Q(x) = \mathrm{e}^{2x} \quad n = 3 \quad v = y^{-2} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{2}y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} + (1-n)P(x)v &= (1-n)Q(x) \\
\frac{\mathrm{d}v}{\mathrm{d}x} + 2v &= -2\mathrm{e}^{2x} \\
\mu(x) &= \mathrm{e}^{\int 2\,\mathrm{d}x} = \mathrm{e}^{2x} \\
\mathrm{e}^{2x}\frac{\mathrm{d}v}{\mathrm{d}x} + 2v\mathrm{e}^{2x} &= -2\mathrm{e}^{4x} \\
\int \left(\mathrm{e}^{2x}\frac{\mathrm{d}v}{\mathrm{d}x} + 2v\mathrm{e}^{2x}\right)\mathrm{d}x &= \int -2\mathrm{e}^{4x}\,\mathrm{d}x \\
v\mathrm{e}^{2x} &= \frac{-2}{4}\mathrm{e}^{4x} + c \\
\frac{\mathrm{e}^{2x}}{y^2} &= \frac{-\mathrm{e}^{4x}}{2} + c
\end{aligned}$$

Exercise 23

Use the method discussed under "Bernoulli Equations" to solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x} - x^2y^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x} = -x^2y^2$$
$$v = y^{-1} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y^2}\frac{2y}{x} = -\frac{1}{y^2}x^2y^2$$
$$\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{xy} = -x^2$$
$$-\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{2v}{x} = -x^2$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{2v}{x} = x^2$$
$$\mu(x) = \mathrm{e}^{\int \frac{2}{x}\ \mathrm{d}x} = \mathrm{e}^{2\ln(x)} = x^2$$
$$x^2\frac{\mathrm{d}v}{\mathrm{d}x} + 2vx = x^4$$
$$\int \left(x^2\frac{\mathrm{d}v}{\mathrm{d}x} + 2vx\right)\ \mathrm{d}x = \int x^4\ \mathrm{d}x$$
$$vx^2 = \frac{x^5}{5} + c$$
$$\frac{x^2}{y} = \frac{x^5}{5} + c$$

Exercise 24

Use the method discussed under "Bernoulli Equations" to solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x-2} = 5(x-2)y^{\frac{1}{2}}$$
$$v = y^{\frac{1}{2}} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{\sqrt{y}}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{1}{\sqrt{y}}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{\sqrt{y}}\frac{y}{x-2} = \frac{1}{\sqrt{y}}5(x-2)\sqrt{y}$$
$$\frac{1}{\sqrt{y}}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\sqrt{y}}{x-2} = 5x - 10$$
$$-\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{v}{x-2} = 5x - 10$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v}{x-2} = 10 - 5x$$
$$\mu(x) = \mathrm{e}^{\int \frac{-1}{x-2}\ \mathrm{d}x} = \mathrm{e}^{-\ln(x-2)} = \frac{1}{x-2}$$
$$(x-2)\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v}{(x-2)^2} = \frac{10-5x}{x-2}$$

$$\int \left((x-2)\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v}{(x-2)^2} \right) \mathrm{d}x = \int \frac{10-5x}{x-2}\ \mathrm{d}x$$
$$v(x-2) = -5x + c$$
$$(x-2)\sqrt{y} = -5x + c$$

Exercise 25

Use the method discussed under "Bernoulli Equations" to solve:

$$\frac{\mathrm{d}x}{\mathrm{d}t} + tx^3 + \frac{x}{t} = 0$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{t} = -x^3$$
$$v = x^{-2} \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{2}x^{-3}$$
$$\frac{1}{x^3}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{x^3}\frac{x}{t} = -\frac{1}{x^3}x^3$$
$$\frac{1}{x^3}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{x^2 t} = -1$$
$$-\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{t} = -1$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{v}{t} = 1$$
$$\mu(t) = \mathrm{e}^{\int -\frac{1}{t}\ \mathrm{d}t} = \mathrm{e}^{-\ln(t)} = \frac{1}{t}$$
$$\frac{1}{t}\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{v}{t^2} = \frac{1}{t}$$
$$\int \left(\frac{1}{t}\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{v}{t^2} \right) \mathrm{d}t = \int \frac{1}{t}\ \mathrm{d}t$$
$$\frac{v}{t} = \ln(t) + c$$
$$\frac{1}{tx^2} = \ln(t) + c$$

Exercise 26

Use the method discussed under "Bernoulli Equations" to solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = \mathrm{e}^x y^{-2}$$
$$v = y^3 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y(y^2) = \mathrm{e}^x y^{-2}(y^2)$$
$$\frac{1}{3}\frac{\mathrm{d}v}{\mathrm{d}x} + v = \mathrm{e}^x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} + 3v = 3\mathrm{e}^x$$
$$\mu(x) = \mathrm{e}^{\ln 3\ \mathrm{d}x} = \mathrm{e}^{3x}$$
$$\mathrm{e}^{3x}\frac{\mathrm{d}v}{\mathrm{d}x} + 3v\mathrm{e}^{3x} = 3\mathrm{e}^{4x}$$

$$\begin{aligned}
\int \left(e^{3x} \frac{dv}{dx} + 3ve^{3x} \right) dx &= \int 3e^{4x} \, dx \\
ve^{3x} &= \frac{3}{4} e^{4x} + c \\
y^3 e^{3x} &= \frac{3}{4} e^{4x} + c
\end{aligned}$$

Exercise 27

Use the method discussed under "Bernoulli Equations" to solve:

$$\begin{aligned}
\frac{dr}{d\theta} &= \frac{r^2 + 2r\theta}{\theta^2} \\
&= \frac{r^2}{\theta^2} + \frac{2r}{\theta} \\
\frac{dr}{d\theta} - \frac{2r}{\theta} &= \frac{r^2}{\theta^2} \\
v = r^{-1} \quad \frac{dv}{d\theta} &= -\frac{1}{r^2} \frac{dr}{d\theta} \\
\frac{1}{r^2} \frac{dr}{d\theta} - \frac{1}{r^2} \frac{2r}{\theta} &= \frac{1}{r^2} \frac{r^2}{\theta^2} \\
-\frac{dv}{d\theta} - \frac{2v}{\theta} &= \frac{1}{\theta^2} \\
\frac{dv}{d\theta} + \frac{2v}{\theta} &= -\frac{1}{\theta^2} \\
\mu(\theta) &= e^{\int \frac{2}{\theta} \, d\theta} = e^{2\ln(\theta)} = \theta^2 \\
\theta^2 \frac{dv}{d\theta} + 2v\theta &= -1 \\
\int \left(\theta^2 \frac{dv}{d\theta} + 2v\theta \right) d\theta &= \int -1 \, d\theta \\
v\theta^2 &= -\theta + c \\
\frac{\theta^2}{r} &= -\theta + c
\end{aligned}$$

Exercise 28

Use the method discussed under "Bernoulli Equations" to solve:

$$\begin{aligned}
\frac{dy}{dx} + y^3 x + y &= 0 \\
\frac{dy}{dx} + y &= -xy^3 \\
v = y^{-2} \quad \frac{dv}{dx} &= -\frac{1}{2} y^{-3} \frac{dy}{dx} \\
\frac{1}{y^3} \frac{dy}{dx} + \frac{1}{y^3} y &= \frac{1}{y^3} - xy^3 \\
-2\frac{dv}{dx} + v &= -x
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v}{2} &= \frac{x}{2} \\
\mu(x) &= \mathrm{e}^{\int -\frac{1}{2}\ \mathrm{d}x} = \mathrm{e}^{-\frac{x}{2}} \\
\mathrm{e}^{-\frac{x}{2}}\frac{\mathrm{d}v}{\mathrm{d}x} - \mathrm{e}^{-\frac{x}{2}}\frac{v}{2} &= \frac{x\mathrm{e}^{\frac{x}{2}}}{2} \\
\int \left(\mathrm{e}^{-\frac{x}{2}}\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{v\mathrm{e}^{-\frac{x}{2}}}{2}\right)\ \mathrm{d}x &= \int \frac{x\mathrm{e}^{-\frac{x}{2}}}{2}\ \mathrm{d}x \\
v\mathrm{e}^{-\frac{x}{2}} &= -\mathrm{e}^{-\frac{x}{2}}(x-2) + c \\
\frac{1}{y^2} &= 2 - x + c
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech