

# Differential Equations: Homework 2

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## Section 1.3

### Exercise 1

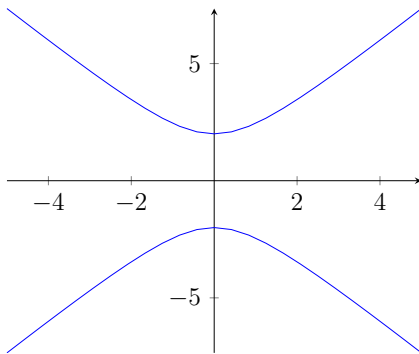
The direction field for  $\frac{dy}{dx} = \frac{4x}{y}$  is shown. Verify that the straight lines  $y = \pm 2x$  are solution curves, provided  $x \neq 0$ .

$$y = 2x$$
$$\frac{dy}{dx} = 2 = \frac{4x}{y} = \frac{4x}{2x} = 2$$

$$y = -2x$$
$$\frac{dy}{dx} = -2 = \frac{4x}{y} = \frac{4x}{-2x} = -2$$

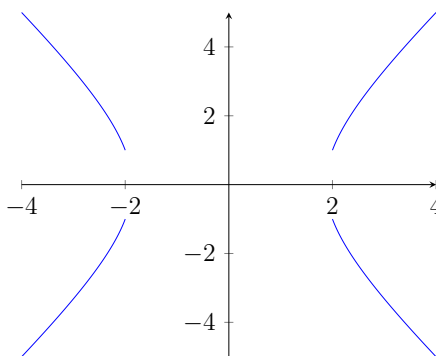
Sketch the solution curve with initial condition  $y(0) = 2$ .

$$\frac{dy}{dx} = \frac{4x}{y}$$
$$y \, dy = 4x \, dx$$
$$\int y \, dy = \int 4x \, dx$$
$$y^2 = 2x^2 + c$$
$$y(0) = 2 \quad c = 4$$
$$y^2 = 2x^2 + 4$$



Sketch the solution curve with initial condition  $y(2) = 1$ .

$$\begin{aligned} \frac{dy}{dx} &= \frac{4x}{y} \\ y \, dy &= 4x \, dx \\ \int y \, dy &= \int 4x \, dx \\ y^2 &= 2x^2 + c \\ y(2) = 1 & \quad c = -7 \\ y^2 &= 2x^2 - 7 \end{aligned}$$



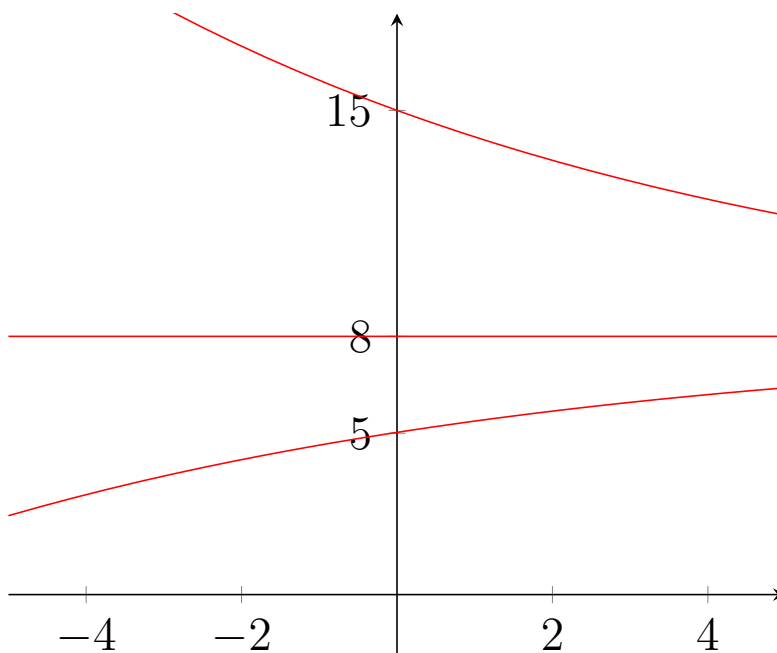
What can you say about the behavior of the above solutions as  $x \rightarrow +\infty$ ? How about  $x \rightarrow -\infty$ ? As  $x$  approaches positive/negative infinity, the solutions become more and more vertical. This makes sense since the slope approaches positive and negative infinity.

### Exercise 3

A model for the velocity  $v$  at time  $t$  of a certain object falling under the influence of gravity in a viscous medium is given by the equation

$$\frac{dv}{dt} = 1 - \frac{v}{8}$$

From the direction field, sketch the solutions with the initial conditions  $v(0) = 5, 8, 15$ . Why is the value  $v = 8$  called the “terminal velocity”?

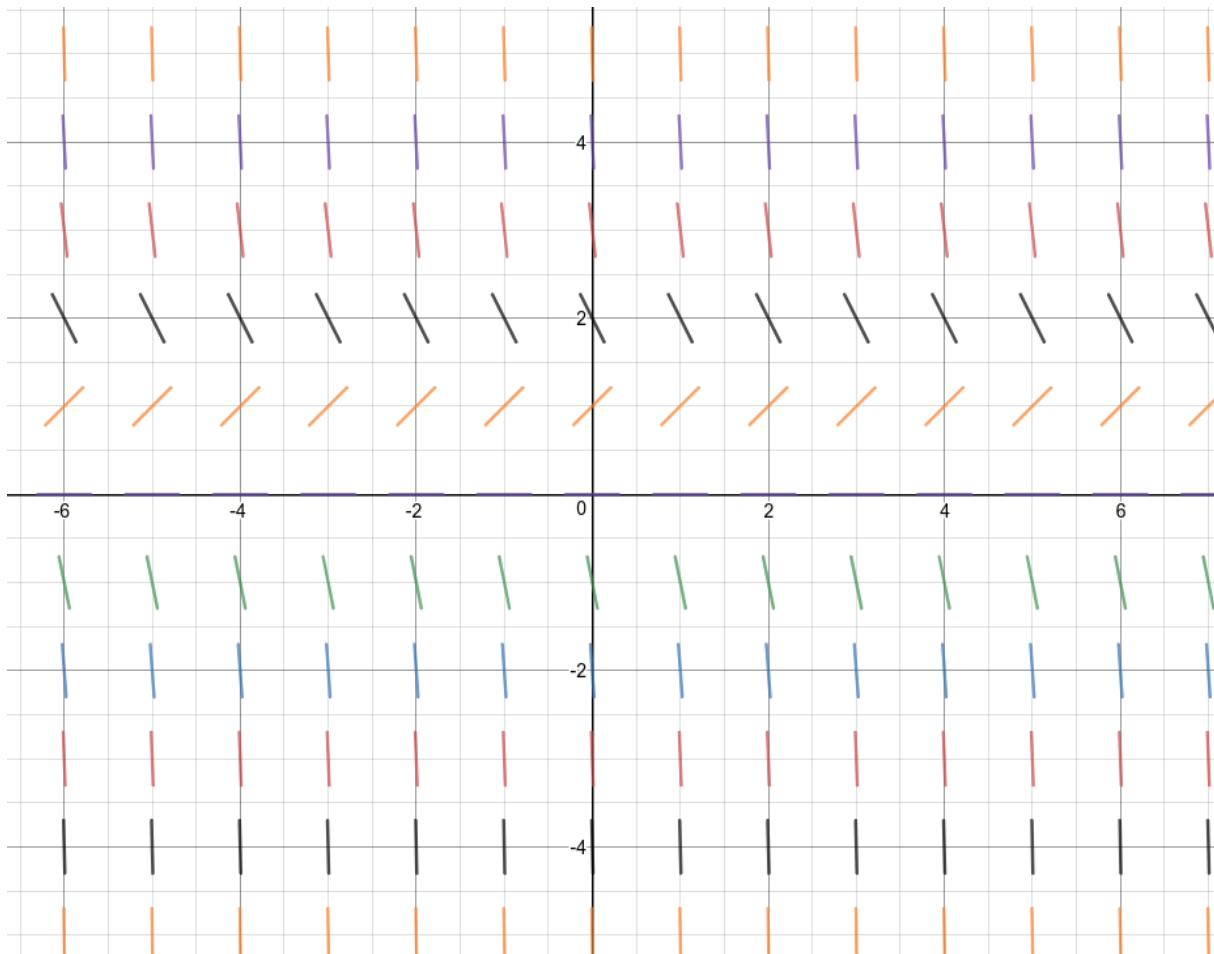


$v = 8$  is called the terminal velocity because no object can cross that line from either side. All objects will eventually accelerate or decelerate to  $v = 8$  from any initial velocity. It represents the maximum velocity that an object can freefall at in the viscous medium.

### Exercise 5

The logistic equation for the population (in thousands) of a certain species is given by:

$$\frac{dp}{dt} = 3p - 2p^2$$



If the initial population is 3000, what can you say about the limiting population  $\lim_{t \rightarrow +\infty} p(t)$ ?

$$\lim_{t \rightarrow +\infty} p(t) = 1500$$

If  $p(0) = 0.8$ , what is  $\lim_{t \rightarrow +\infty} p(t)$ ?

$$\lim_{t \rightarrow +\infty} p(t) = 1500$$

Can a population of 2000 ever decline to 800?

No, the population will stop decreasing at 1500.

## Section 1.4

### Exercise 5

Use Euler's method with step size  $h = 0.1$  to approximate the solution to the initial value problem

$$y' = x - y^2 \quad y(1) = 0$$

at the points  $x = 1.1, 1.2, 1.3, 1.4, 1.5$ .

$x_0 = 1$	$y_0 = 1$
$x_1 = 1.1$	$y_1 = 1 + 0.1(1 - 1^2) = 1$
$x_2 = 1.2$	$y_2 = 1 + 0.1(1.1 - 1^2) = 1.01$
$x_3 = 1.3$	$y_3 = 1.01 + 0.1(1.2 - 1.01^2) = 1.028$
$x_4 = 1.4$	$y_4 = 1.028 + 0.1(1.3 - 1.028^2) = 1.052$
$x_5 = 1.5$	$y_5 = 1.052 + 0.1(1.4 - 1.052^2) = 1.081$

## Section 2.2

### Exercise 3

Determine whether the given differential equation is separable.

$$\begin{aligned}\frac{ds}{dt} &= t \ln(s^{2t}) + 8t^2 \\ &= 2t^2 \ln(s) + 8t^2 \\ &= 2t^2(\ln(s) + 4) \\ \frac{1}{\ln(s) + 4} ds &= 2t^2 dt\end{aligned}$$

### Exercise 7

Solve the equation:

$$\begin{aligned}x dy &= \frac{1}{y^3} \\ y^3 dy &= \frac{1}{x} dx \\ \int y^3 dy &= \int \frac{1}{x} dx \\ \frac{y^4}{4} &= \ln|x| + c\end{aligned}$$

### Exercise 9

Solve the equation:

$$\begin{aligned}\frac{dx}{dt} &= \frac{t}{xe^{t+2x}} \\ xe^{t+2x} dx &= t dt \\ xe^t e^{2x} dx &= t dt \\ xe^{2x} dx &= \frac{t}{e^t} dt \\ \int xe^{2x} dx &= \int \frac{t}{e^t} dt \\ xe^x - e^x &= -te^{-t} - e^{-t} + c\end{aligned}$$

**Exercise 12**

Solve the equation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sec^2(y)}{1+x^2} \\ \frac{1}{\sec^2(y)} dy &= \frac{1}{1+x^2} dx \\ \int \cos^2(y) dy &= \int \frac{1}{1+x^2} dx \\ \frac{x}{2} + \frac{\sin(y)\cos(y)}{2} &= \tan^{-1}(x) + c\end{aligned}$$

**Exercise 15**

Solve the equation:

$$\begin{aligned}(x + xy^2) dx + e^{x^2} y dy &= 0 \\ x(1 + y^2) dx &= -e^{x^2} y dy \\ -\frac{x}{e^{x^2}} dx &= \frac{y}{1+y^2} dy \\ -\int \frac{x}{e^{x^2}} dx &= \int \frac{y}{1+y^2} dy \\ \frac{-1}{2e^{x^2}} + c &= \frac{\ln|y^2+1|}{2}\end{aligned}$$

**Exercise 17**Solve the initial value problem  $\frac{dy}{dx} = (1+y^2)\tan(x)$  with  $y(0) = \sqrt{3}$ :

$$\begin{aligned}\frac{dy}{dx} &= (1+y^2)\tan(x) \\ \frac{1}{1+y^2} dy &= \tan(x) dx \\ \int \frac{1}{1+y^2} dy &= \int \tan(x) dx \\ \tan^{-1}(y) &= -\ln|\cos(x)| + c \\ \tan^{-1}(\sqrt{3}) &= -\ln|\cos(0)| + c \\ \frac{\pi}{3} &= c \\ \tan^{-1}(y) &= -\ln|\cos(x)| + \frac{\pi}{3}\end{aligned}$$

**Exercise 19**

Solve the initial value problem  $\frac{1}{2} \frac{dy}{dx} = \sqrt{y+1} \cos(x)$  with  $y(\pi) = 0$ :

$$\begin{aligned} \frac{1}{2} \frac{dy}{dx} &= \sqrt{y+1} \cos(x) \\ \frac{1}{2\sqrt{y+1}} dy &= \cos(x) dx \\ \int \frac{1}{2\sqrt{y+1}} dy &= \int \cos(x) dx \\ \sqrt{y+1} &= \sin(x) + c \\ \sqrt{1} &= \sin(\pi) + c \\ c &= 1 \\ y &= (\sin(x) + 1)^2 - 1 \end{aligned}$$

**Exercise 21**

Solve the initial value problem  $\frac{1}{\theta} \frac{dy}{d\theta} = \frac{y \sin \theta}{y^2+1}$  with  $y(\pi) = 1$ :

$$\begin{aligned} \frac{1}{\theta} \frac{dy}{d\theta} &= \frac{y \sin \theta}{y^2+1} \\ \frac{y^2+1}{y} dy &= \theta \sin \theta d\theta \\ \int \frac{y^2+1}{y} dy &= \int \theta \sin \theta d\theta \\ \frac{y^2}{2} + \ln |y| &= \sin \theta - \theta \cos \theta + c \\ \frac{1^2}{2} + \ln |1| &= \sin \pi - \pi \cos \pi + c \\ \frac{1}{2} + 0 &= 0 + \pi + c \\ c &= \frac{1}{2} - \pi \\ \frac{y^2}{2} + \ln |y| &= \sin \theta + \theta \cos \theta + \frac{1}{2} - \pi \end{aligned}$$

**Exercise 23**

Solve the initial value problem  $\frac{dy}{dt} = 2t \cos^2(y)$  with  $y(0) = \frac{\pi}{4}$ .

$$\begin{aligned}\frac{dy}{dt} &= 2t \cos^2(y) \\ \sec^2(y) \, dy &= 2t \, dt \\ \int \sec^2(y) \, dy &= \int 2t \, dt \\ \tan(y) &= t^2 + c \\ \tan\left(\frac{\pi}{4}\right) &= 0 + c \\ c &= 1 \\ \tan(y) &= t^2 + 1 \\ y &= \tan^{-1}(t^2 + 1)\end{aligned}$$

**Exercise 24**

Solve the initial value problem  $\frac{dy}{dx} = 8x^3 e^{-2y}$  with  $y(1) = 0$ .

$$\begin{aligned}\frac{dy}{dx} &= 8x^3 e^{-2y} \\ e^{2y} \, dy &= 8x^3 \, dx \\ \int e^{2y} \, dy &= \int 8x^3 \, dx \\ \frac{e^{2y}}{2} &= 2x^4 + c \\ \frac{e^0}{2} &= 2(1^4) + c \\ c &= -\frac{3}{2} \\ e^{2y} &= 4x^4 - 3 \\ y &= \frac{\ln(4x^4 - 3)}{2}\end{aligned}$$



**Exercise 26**

Solve the initial value problem  $\sqrt{y} \, dx + (1 + x) \, dy = 0$  with  $y(0) = 1$ .

$$\begin{aligned}\sqrt{y} \, dx + (1 + x) \, dy &= 0 \\(1 + x) \, dy &= -\sqrt{y} \, dx \\-y^{-\frac{1}{2}} \, dy &= \frac{1}{1 + x} \, dx \\-\int y^{-\frac{1}{2}} \, dy &= \int \frac{1}{1 + x} \, dx \\-2\sqrt{y} &= \ln |1 + x| + c \\-2\sqrt{1} &= \ln |1| + c \\-2 &= c \\-2\sqrt{y} &= \ln |1 + x| - 2 \\y &= \left( \frac{\ln |1 + x| - 2}{-2} \right)^2\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)