Differential Equations: Homework 2

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Section 1.3

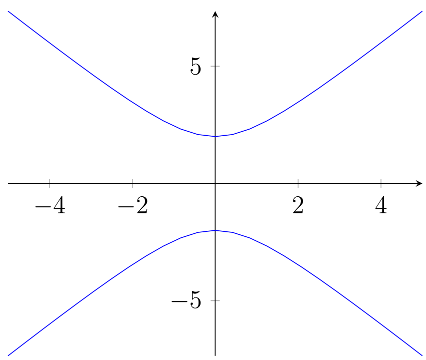
Exercise 1

The direction field for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{y}$ is shown. Verify that the straight lines $y = \pm 2x$ are solution curves, provided $x \neq 0$.

$$\begin{aligned}
y &= 2x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2 = \frac{4x}{y} = \frac{4x}{2x} = 2 \\
y &= -2x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -2 = \frac{4x}{y} = \frac{4x}{-2x} = -2
\end{aligned}$$

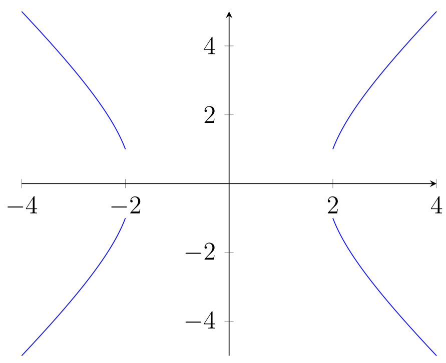
Sketch the solution curve with initial condition $y(0) = 2$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4x}{y} \\
y\ \mathrm{d}y &= 4x\ \mathrm{d}x \\
\int y\ \mathrm{d}y &= \int 4x\ \mathrm{d}x \\
y^2 &= 2x^2 + c \\
y(0) = 2 &\quad c = 4 \\
y^2 &= 2x^2 + 4
\end{aligned}$$



Sketch the solution curve with initial condition $y(2) = 1$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4x}{y} \\
y\ \mathrm{d}y &= 4x\ \mathrm{d}x \\
\int y\ \mathrm{d}y &= \int 4x\ \mathrm{d}x \\
y^2 &= 2x^2 + c \\
y(2) = 1 &\quad c = -7 \\
y^2 &= 2x^2 - 7
\end{aligned}$$



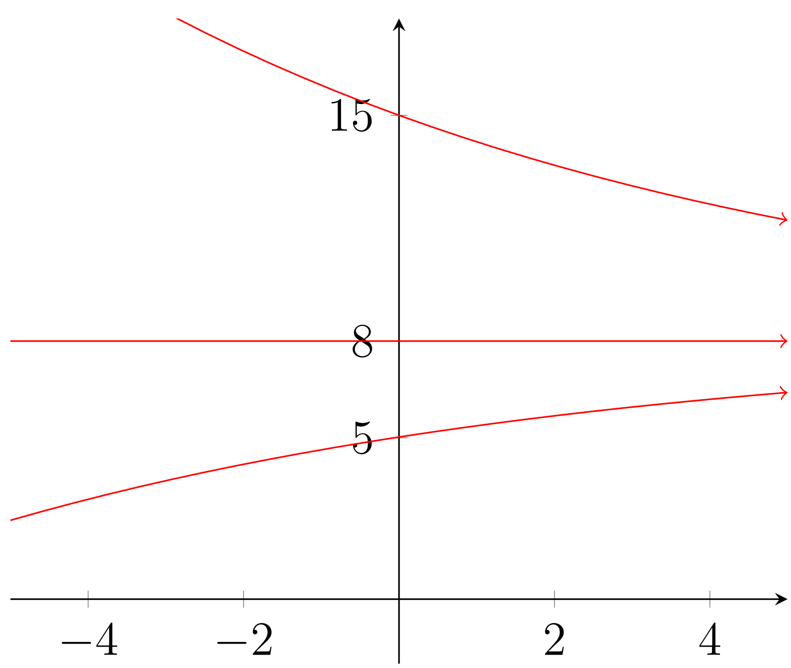
What can you say about the behavior of the above solutions as $x \to +\infty$? How about $x \to -\infty$?
As x approaches positive/negative infinity, the solutions become more and more vertical. This makes sense since the slope approaches positive and negative infinity.

Exercise 3

A model for the velocity v at time t of a certain object falling under the influence of gravity in a viscous medium is given by the equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 1 - \frac{v}{8}$$

From the direction field, sketch the solutions with the initial conditions $v(0) = 5, 8, 15$. Why is the value $v = 8$ called the "terminal velocity"?

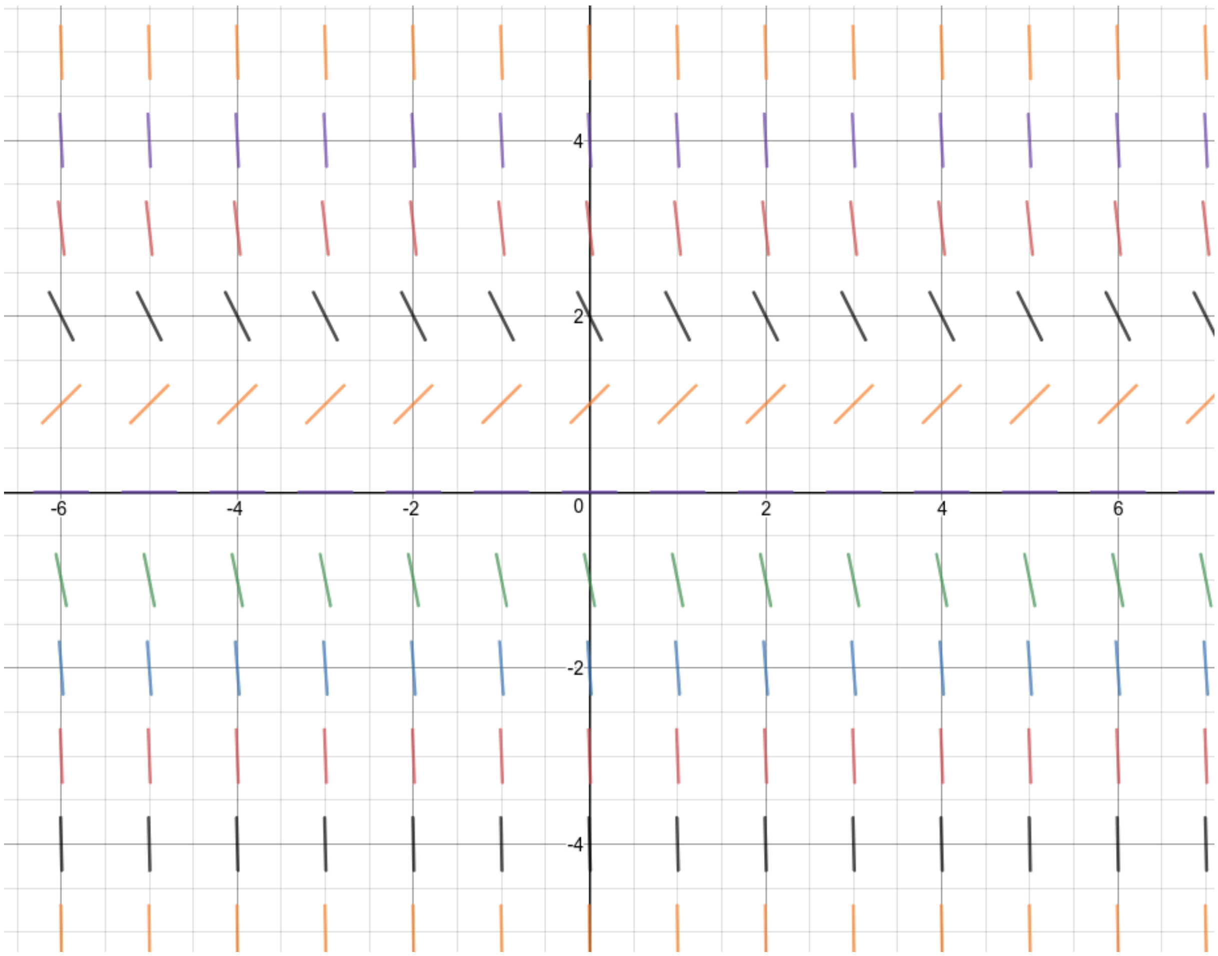


$v = 8$ is called the terminal velocity because no object can cross that line from either side. All objects will eventually accelerate or decelerate to $v = 8$ from any initial velocity. It represents the maximum velocity that an object can freefall at in the viscous medium.

Exercise 5

The logistic equation for the population (in thousands) of a certain species is given by:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 3p - 2p^2$$



If the initial population is 3000, what can you say about the limiting population $\lim_{t\to+\infty} p(t)$?

$$\lim_{t\to+\infty} p(t) = 1500$$

If $p(0) = 0.8$, what is $\lim_{t\to+\infty} p(t)$?

$$\lim_{t\to+\infty} p(t) = 1500$$

Can a population of 2000 ever decline to 800?
No, the population will stop decreasing at 1500.

Section 1.4

Exercise 5

Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem

$$y' = x - y^2 \quad y(1) = 0$$

at the points $x = 1.1, 1.2, 1.3, 1.4, 1.5$.

$x_0 = 1$	$y_0 = 1$
$x_1 = 1.1$	$y_1 = 1 + 0.1(1 - 1^2) = 1$
$x_2 = 1.2$	$y_2 = 1 + 0.1(1.1 - 1^2) = 1.01$
$x_3 = 1.3$	$y_3 = 1.01 + 0.1(1.2 - 1.01^2) = 1.028$
$x_4 = 1.4$	$y_4 = 1.028 + 0.1(1.3 - 1.028^2) = 1.052$
$x_5 = 1.5$	$y_5 = 1.052 + 0.1(1.4 - 1.052^2) = 1.081$

Section 2.2

Exercise 3

Determine whether the given differential equation is separable.

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}t} &= t\ln(s^{2t}) + 8t^2 \\
&= 2t^2\ln(s) + 8t^2 \\
&= 2t^2(\ln(s) + 4) \\
\frac{1}{\ln(s)+4}\ \mathrm{d}s &= 2t^2\ \mathrm{d}t
\end{aligned}$$

Exercise 7

Solve the equation:

$$\begin{aligned}
x\ \mathrm{d}yx &= \frac{1}{y^3} \\
y^3\ \mathrm{d}y &= \frac{1}{x}\ \mathrm{d}x \\
\int y^3\ \mathrm{d}y &= \int \frac{1}{x}\ \mathrm{d}x \\
\frac{y^4}{4} &= \ln|x| + c
\end{aligned}$$

Exercise 9

Solve the equation:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{t}{x\mathrm{e}^{t+2x}} \\
x\mathrm{e}^{t+2x}\ \mathrm{d}x &= t\ \mathrm{d}t \\
x\mathrm{e}^{t}\mathrm{e}^{2x}\ \mathrm{d}x &= t\ \mathrm{d}t \\
x\mathrm{e}^{2x}\ \mathrm{d}x &= \frac{t}{\mathrm{e}^t}\ \mathrm{d}t \\
\int x\mathrm{e}^{2x}\ \mathrm{d}x &= \int \frac{t}{\mathrm{e}^t}\ \mathrm{d}t \\
x\mathrm{e}^{x} - \mathrm{e}^{x} &= -t\mathrm{e}^{-t} - \mathrm{e}^{-t} + c
\end{aligned}$$

Exercise 12

Solve the equation:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sec^2(y)}{1+x^2} \\
\frac{1}{\sec^2(y)}\ \mathrm{d}y &= \frac{1}{1+x^2}\ \mathrm{d}x \\
\int \cos^2(y)\ \mathrm{d}y &= \int \frac{1}{1+x^2}\ \mathrm{d}x \\
\frac{x}{2} + \frac{\sin(y)\cos(y)}{2} &= \tan^{-1}(x) + c
\end{aligned}$$

Exercise 15

Solve the equation:

$$\begin{aligned}
(x + xy^2)\ \mathrm{d}x + \mathrm{e}^{x^2} y\ \mathrm{d}y &= 0 \\
x(1+y^2)\ \mathrm{d}x &= -\mathrm{e}^{x^2} y\ \mathrm{d}y \\
-\frac{x}{\mathrm{e}^{x^2}}\ \mathrm{d}x &= \frac{y}{1+y^2}\ \mathrm{d}y \\
-\int \frac{x}{\mathrm{e}^{x^2}}\ \mathrm{d}x &= \int \frac{y}{1+y^2}\ \mathrm{d}y \\
\frac{-1}{2\mathrm{e}^{x^2}} + c &= \frac{\ln|y^2+1|}{2}
\end{aligned}$$

Exercise 17

Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = (1+y^2)\tan(x)$ with $y(0) = \sqrt{3}$:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= (1+y^2)\tan(x) \\
\frac{1}{1+y^2}\ \mathrm{d}y &= \tan(x)\ \mathrm{d}x \\
\int \frac{1}{1+y^2}\ \mathrm{d}y &= \int \tan(x)\ \mathrm{d}x \\
\tan^{-1}(y) &= -\ln|\cos(x)| + c \\
\tan^{-1}(\sqrt{3}) &= -\ln|\cos(0)| + c \\
\frac{\pi}{3} &= c \\
\tan^{-1}(y) &= -\ln|\cos(x)| + \frac{\pi}{3}
\end{aligned}$$

Exercise 19

Solve the initial value problem $\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y+1}\cos(x)$ with $y(\pi) = 0$:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{y+1}\cos(x) \\
\frac{1}{2\sqrt{y+1}}\ \mathrm{d}y &= \cos(x)\ \mathrm{d}x \\
\int \frac{1}{2\sqrt{y+1}}\ \mathrm{d}y &= \int \cos(x)\ \mathrm{d}x \\
\sqrt{y+1} &= \sin(x) + c \\
\sqrt{1} &= \sin(\pi) + c \\
c &= 1 \\
y &= (\sin(x) + 1)^2 - 1
\end{aligned}$$

Exercise 21

Solve the initial value problem $\frac{1}{\theta}\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{y\sin\theta}{y^2+1}$ with $y(\pi) = 1$:

$$\begin{aligned}
\frac{1}{\theta}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{y\sin\theta}{y^2+1} \\
\frac{y^2+1}{y}\ \mathrm{d}y &= \theta\sin\theta\ \mathrm{d}\theta \\
\int \frac{y^2+1}{y}\ \mathrm{d}y &= \int \theta\sin\theta\ \mathrm{d}\theta \\
\frac{y^2}{2} + \ln|y| &= \sin\theta - \theta\cos\theta + c \\
\frac{1^2}{2} + \ln|1| &= \sin\pi - \pi\cos\pi + c \\
\frac{1}{2} + 0 &= 0 + \pi + c \\
c &= \frac{1}{2} - \pi \\
\frac{y^2}{2} + \ln|y| &= \sin\theta + \theta\cos\theta + \frac{1}{2} - \pi
\end{aligned}$$

Exercise 23

Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}t} = 2t\cos^2(y)$ with $y(0) = \frac{\pi}{4}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= 2t\cos^2(y) \\
\sec^2(y)\ \mathrm{d}y &= 2t\ \mathrm{d}t \\
\int \sec^2(y)\ \mathrm{d}y &= \int 2t\ \mathrm{d}t \\
\tan(y) &= t^2 + c \\
\tan(\frac{\pi}{4}) &= 0 + c \\
c &= 1 \\
\tan(y) &= t^2 + 1 \\
y &= \tan^{-1}(t^2 + 1)
\end{aligned}$$

Exercise 24

Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^3\mathrm{e}^{-2y}$ with $y(1) = 0$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 8x^3\mathrm{e}^{-2y} \\
\mathrm{e}^{2y}\ \mathrm{d}y &= 8x^3\ \mathrm{d}x \\
\int \mathrm{e}^{2y}\ \mathrm{d}y &= \int 8x^3\ \mathrm{d}x \\
\frac{\mathrm{e}^{2y}}{2} &= 2x^4 + c \\
\frac{\mathrm{e}^{0}}{2} &= 2(1^4) + c \\
c &= -\frac{3}{2} \\
\mathrm{e}^{2y} &= 4x^4 - 3 \\
y &= \frac{\ln(4x^4 - 3)}{2}
\end{aligned}$$

Exercise 26

Solve the initial value problem $\sqrt{y}\ \mathrm{d}x + (1+x)\ \mathrm{d}y = 0$ with $y(0) = 1$.

$$
\begin{aligned}
\sqrt{y}\ \mathrm{d}x + (1+x)\ \mathrm{d}y &= 0 \\
(1+x)\ \mathrm{d}y &= -\sqrt{y}\ \mathrm{d}x \\
-y^{-\frac{1}{2}}\ \mathrm{d}y &= \frac{1}{1+x}\ \mathrm{d}x \\
-\int y^{-\frac{1}{2}}\ \mathrm{d}y &= \int \frac{1}{1+x}\ \mathrm{d}x \\
-2\sqrt{y} &= \ln|1+x| + c \\
-2\sqrt{1} &= \ln|1| + c \\
-2 &= c \\
-2\sqrt{y} &= \ln|1+x| - 2 \\
y &= \left(\frac{\ln|1+x| - 2}{-2}\right)^2
\end{aligned}
$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech