

Differential Equations: Homework 1

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Exercise 1

Show that $\phi(x) = x^2$ is an explicit solution to

$$x \frac{dy}{dx} = 2y$$

on the interval $(-\infty, \infty)$.

$$\begin{aligned}\phi(x) &= x^2 \\ \phi'(x) &= 2x \\ x \frac{dy}{dx} &= 2y \\ x(2x) &= 2(x^2) \\ 2x^2 &= 2x^2\end{aligned}$$

Show that $\phi(x) = e^x - x$ is an explicit solution to

$$\frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1$$

on the interval $(-\infty, \infty)$.

$$\begin{aligned}\phi(x) &= e^x - x \\ \phi'(x) &= e^x - 1 \\ \frac{dy}{dx} + y^2 &= e^{2x} + (1 - 2x)e^x + x^2 - 1 \\ e^x - 1 + (e^x - x)^2 &= e^{2x} + (1 - 2x)e^x + x^2 - 1 \\ e^x - 1 + (e^{2x} - 2xe^x + x^2) &= e^{2x} + (1 - 2x)e^x + x^2 - 1 \\ e^{2x} + e^x - 2xe^x + x^2 - 1 &= e^{2x} + (1 - 2x)e^x + x^2 - 1 \\ e^{2x} + (1 - 2x)e^x + x^2 - 1 &= e^{2x} + (1 - 2x)e^x + x^2 - 1\end{aligned}$$

Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to

$$x^2 \frac{d^2y}{dx^2} = 2y$$

on the interval $(0, \infty)$.

$$\begin{aligned}\phi(x) &= x^2 - \frac{1}{x} \\ \phi'(x) &= 2x + x^{-2} \\ \phi''(x) &= 2 - \frac{2}{x^3} \\ x^2 \frac{d^2y}{dx^2} &= 2y \\ x^2(2 - \frac{2}{x^3}) &= 2(x^2 - \frac{1}{x}) \\ 2x^2 - \frac{2}{x} &= 2x^2 - \frac{2}{x}\end{aligned}$$

Exercise 2

Show that $y^2 + x - 3 = 0$ is an implicit solution to $\frac{dy}{dx} = \frac{-1}{2y}$ on the interval $(-\infty, 3)$.

$$\begin{aligned}y^2 + x - 3 &= 0 \\2y \frac{dy}{dx} + 1 &= 0 \\2y \frac{dy}{dx} &= -1 \\\frac{dy}{dx} &= \frac{-1}{2y}\end{aligned}$$

Show that $xy^3 - xy^3 \sin(x) = 1$ is an implicit solution to

$$\frac{dy}{dx} = \frac{(x \cos(x) + \sin(x) - 1)y}{3(x - x \sin(x))}$$

on the interval $(0, \frac{\pi}{2})$.

$$\begin{aligned}xy^3 - xy^3 \sin(x) &= 1 \\\frac{d}{dx}(xy^3) - \left[\frac{d}{dx}(xy^3) \sin(x) + xy^3 \frac{d}{dx}(\sin(x)) \right] &= 0 \\(1 - \sin(x)) \frac{d}{dx}(xy^3) - xy^3 \cos(x) &= 0 \\(1 - \sin(x))(3xy^2 \frac{dy}{dx} + y^3) - xy^3 \cos(x) &= 0 \\3xy^2 \frac{dy}{dx} + y^3 - 3xy^2 \sin(x) \frac{dy}{dx} - y^3 \sin(x) - xy^3 \cos(x) &= 0 \\3xy^2 \frac{dy}{dx} - 3xy^2 \sin(x) \frac{dy}{dx} &= xy^3 \cos(x) + y^3 \sin(x) - y^3 \\\frac{dy}{dx} 3y^2(x - x \sin(x)) &= (x \cos(x) + \sin(x) - 1)y^3 \\\frac{dy}{dx} &= \frac{(x \cos(x) + \sin(x) - 1)y^3}{3y^2(x - x \sin(x))} \\\frac{dy}{dx} &= \frac{(x \cos(x) + \sin(x) - 1)y}{3(x - x \sin(x))}\end{aligned}$$

Exercise 3

Determine whether the given function is a solution to the given differential equation.

$$y = \sin(x) + x^2 \quad \frac{d^2y}{dx^2} + y = x^2 + 2$$

$$\begin{aligned}
y &= \sin(x) + x^2 \\
\frac{dy}{dx} &= \cos(x) + 2x \\
\frac{d^2y}{dx^2} &= -\sin(x) + 2 \\
\frac{d^2y}{dx^2} + y &= x^2 + 2 \\
(-\sin(x) + 2) + (\sin(x) + x^2) &= x^2 + 2 \\
x^2 + 2 &= x^2 + 2
\end{aligned}$$

Exercise 6

Determine whether the given function is a solution to the given differential equation.

$$\begin{aligned}
x = \cos(t) \quad \frac{dx}{dt} + tx &= \sin(2t) \\
x &= \cos(2t) \\
\frac{dx}{dt} &= -2 \sin(2t) \\
\frac{dx}{dt} + tx &= \sin(2t) \\
-2 \sin(2t) + t(\cos(2t)) &= \sin(2t) \\
t \cos(2t) &= 3 \sin(2t)
\end{aligned}$$

No solution.

Exercise 7

Determine whether the given function is a solution to the given differential equation.

$$\begin{aligned}
y = e^{2x} - 3e^{-x} \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y &= 0 \\
y &= e^{2x} - 3e^{-x} \\
\frac{dy}{dx} &= 2e^{2x} + 3e^{-x} \\
\frac{d^2y}{dx^2} &= 4e^{2x} - 3e^{-x} \\
\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y &= 0 \\
(4e^{2x} - 3e^{-x}) - (2e^{2x} + 3e^{-x}) - 2(e^{2x} - 3e^{-x}) &= 0 \\
0 &= 0
\end{aligned}$$

Exercise 8

Determine whether the given function is a solution to the given differential equation.

$$y = 3 \sin(2x) + e^{-x} \quad y'' + 4y = 5e^{-x}$$

$$\begin{aligned}
y &= 3 \sin(2x) + e^{-x} \\
y' &= 6 \cos(2x) - e^{-x} \\
y'' &= -12 \sin(2x) + e^{-x} \\
y'' + 4y &= 5e^{-x} \\
(-12 \sin(2x) + e^{-x}) + 4(3 \sin(2x) + e^{-x}) &= 5e^{-x} \\
5e^{-x} &= 5e^{-x}
\end{aligned}$$

Exercise 9

Determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

$$x^2 + y^2 = 4 \quad \frac{dy}{dx} = \frac{x}{y}$$

$$\begin{aligned}
x^2 + y^2 &= 4 \\
2x + 2y \frac{dy}{dx} &= 0 \\
\frac{dy}{dx} 2y &= -2x \\
\frac{dy}{dx} &= \frac{-x}{y}
\end{aligned}$$

Not a solution.

Exercise 10

Determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

$$y - \ln(y) = x^2 + 1 \quad \frac{dy}{dx} = \frac{2xy}{y-1}$$

$$\begin{aligned}
y - \ln(y) &= x^2 + 1 \\
\frac{dy}{dx} - \frac{dy}{dx} \frac{1}{y} &= 2x \\
\frac{dy}{dx} \left(1 - \frac{1}{y}\right) &= 2x \\
\frac{dy}{dx} (y - 1) &= 2xy \\
\frac{dy}{dx} &= \frac{2xy}{y-1}
\end{aligned}$$

Exercise 11

Determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

$$e^{xy} + y = x - 1 \quad \frac{dy}{dx} = \frac{e^{-xy} - y}{e^{-xy} + x}$$

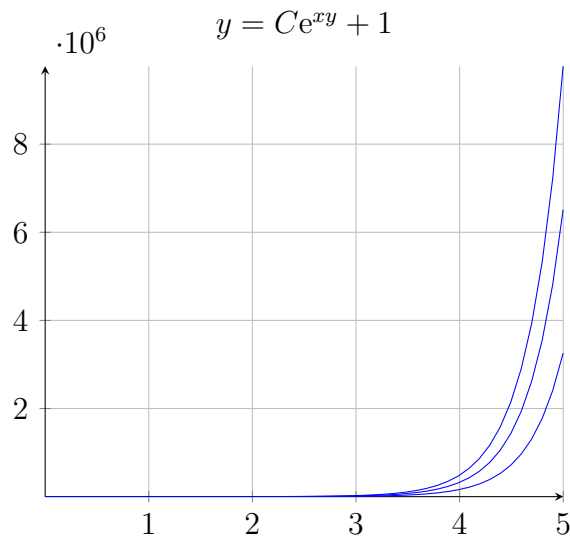
$$\begin{aligned}
e^{xy} + y &= x - 1 \\
e^{xy} \frac{d}{dx}(xy) + \frac{dy}{dx} &= 1 \\
e^{xy} \left(x \frac{dy}{dx} + y \right) + \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} x e^{xy} + y e^{xy} + \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} (1 + x e^{xy}) &= 1 - y e^{xy} \\
\frac{dy}{dx} &= \frac{1 - y e^{xy}}{1 + x e^{xy}}
\end{aligned}$$

Not a solution.

Exercise 17

Show that $\phi(x) = Ce^{3x} + 1$ is a solution to $\frac{dy}{dx} - 3y = -3$ for any choice of the constant C . Thus, $Ce^{3x} + 1$ is a one-parameter family of solutions to the differential equation. Graph several of the solution curves using the same coordinate axes.

$$\begin{aligned}
\phi(x) &= Ce^{3x} + 1 \\
\phi'(x) &= 3Ce^{3x} \\
\frac{dy}{dx} - 3y &= -3 \\
3Ce^{3x} - 3(Ce^{3x} + 1) &= -3 \\
-3 &= -3
\end{aligned}$$



Exercise 19

Show that the equation $(\frac{dy}{dx})^2 + y^2 + 4 = 0$ has no (real-valued) solution.

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2 + y^2 + 4 &= 0 \\ \frac{dy}{dx} &= \pm\sqrt{-y^2 - 4} \\ \frac{dy}{dx} &= \pm\sqrt{-1}\sqrt{y^2 + 4} \\ \frac{dy}{dx} &= \pm\sqrt{y^2 + 4}i\end{aligned}$$

Solutions are imaginary.

Exercise 22

Verify that the function $\phi(x) = c_1e^x + c_2e^{-2x}$ is a solution to the linear equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2 .

$$\begin{aligned}\phi(x) &= c_1e^x + c_2e^{-2x} \\ \phi'(x) &= c_1e^x - 2c_2e^{-2x} \\ \phi''(x) &= c_1e^x + 4c_2e^{-2x} \\ \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y &= 0 \\ (c_1e^x + 4c_2e^{-2x}) + (c_1e^x - 2c_2e^{-2x}) - 2(c_1e^x + c_2e^{-2x}) &= 0 \\ 0 &= 0\end{aligned}$$

Determine c_1 and c_2 so that each of the following initial conditions is satisfied.

$$\begin{aligned}y(0) = 2 \quad y'(0) = 1 \\ y(0) = c_1e^0 + c_2e^0 = 2 \\ c_1 + c_2 = 2 \\ y'(0) = c_1e^0 - 2c_2e^0 = 1 \\ c_1 - 2c_2 = 1 \\ (2 - c_2) - 2c_2 = 1 \\ c_2 = \frac{1}{3} \quad c_1 = \frac{5}{3}\end{aligned}$$

$$\begin{aligned}y(1) = 1 \quad y'(1) = 0 \\ y(1) = c_1e + c_2e^{-2} = 1 \\ y'(1) = c_1e - 2c_2e^{-2} = 0 \\ 3c_2e^{-2} = 1 \\ c_2 = \frac{e^2}{3} \quad c_1 = \frac{2}{3e}\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech