Differential Equations: Homework 1

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Exercise 1

Show that $\phi(x) = x^2$ is an explicit solution to

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$$

on the interval $(-\infty, \infty)$.

$$\begin{aligned}
\phi(x) &= x^2 \\
\phi'(x) &= 2x \\
x\frac{\mathrm{d}y}{\mathrm{d}x} &= 2y \\
x(2x) &= 2(x^2) \\
2x^2 &= 2x^2
\end{aligned}$$

Show that $\phi(x) = \mathrm{e}^x - x$ is an explicit solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = \mathrm{e}^{2x} + (1-2x)\mathrm{e}^x + x^2 - 1$$

on the interval $(-\infty, \infty)$.

$$\begin{aligned}
\phi(x) &= \mathrm{e}^x - x \\
\phi'(x) &= \mathrm{e}^x - 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 &= \mathrm{e}^{2x} + (1-2x)\mathrm{e}^x + x^2 - 1 \\
\mathrm{e}^x - 1 + (\mathrm{e}^x - x)^2 &= \mathrm{e}^{2x} + (1-2x)\mathrm{e}^x + x^2 - 1 \\
\mathrm{e}^x - 1 + (\mathrm{e}^{2x} - 2x\mathrm{e}^x + x^2) &= \mathrm{e}^{2x} + (1-2x)\mathrm{e}^x + x^2 - 1 \\
\mathrm{e}^{2x} + \mathrm{e}^x - 2x\mathrm{e}^x + x^2 - 1 &= \mathrm{e}^{2x} + (1-2x)\mathrm{e}^x + x^2 - 1 \\
\mathrm{e}^{2x} + (1-2x)\mathrm{e}^x + x^2 - 1 &= \mathrm{e}^{2x} + (1-2x)\mathrm{e}^x + x^2 - 1
\end{aligned}$$

Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y$$

on the interval $(0, \infty)$.

$$\begin{aligned}
\phi(x) &= x^2 - \frac{1}{x} \\
\phi'(x) &= 2x + x^{-2} \\
\phi''(x) &= 2 - \frac{2}{x^3} \\
x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2y \\
x^2(2 - \frac{2}{x^3}) &= 2(x^2 - \frac{1}{x}) \\
2x^2 - \frac{2}{x} &= 2x^2 - \frac{2}{x}
\end{aligned}$$

Exercise 2

Show that $y^2 + x - 3 = 0$ is an implicit solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{2y}$ on the interval $(-\infty, 3)$.

$$\begin{aligned} y^2 + x - 3 &= 0 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 1 &= 0 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= -1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-1}{2y} \end{aligned}$$

Show that $xy^3 - xy^3\sin(x) = 1$ is an implicit solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x\cos(x) + \sin(x) - 1)y}{3(x - x\sin(x))}$$

on the interval $(0, \frac{\pi}{2})$.

$$\begin{aligned} xy^3 - xy^3\sin(x) &= 1 \\ \frac{\mathrm{d}}{\mathrm{d}x}(xy^3) - \left[\frac{\mathrm{d}}{\mathrm{d}x}(xy^3)\sin(x) + xy^3\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x))\right] &= 0 \\ (1 - \sin(x))\frac{\mathrm{d}}{\mathrm{d}x}(xy^3) - xy^3\cos(x) &= 0 \\ (1 - \sin(x))(3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3) - xy^3\cos(x) &= 0 \\ 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3 - 3xy^2\sin(x)\frac{\mathrm{d}y}{\mathrm{d}x} - y^3\sin(x) - xy^3\cos(x) &= 0 \\ 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} - 3xy^2\sin(x)\frac{\mathrm{d}y}{\mathrm{d}x} &= xy^3\cos(x) + y^3\sin(x) - y^3 \\ \frac{\mathrm{d}y}{\mathrm{d}x}3y^2(x - x\sin(x)) &= (x\cos(x) + \sin(x) - 1)y^3 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x\cos(x) + \sin(x) - 1)y^3}{3y^2(x - x\sin(x))} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x\cos(x) + \sin(x) - 1)y}{3(x - x\sin(x))} \end{aligned}$$

Exercise 3

Determine whether the given function is a solution to the given differential equation.

$$y = \sin(x) + x^2 \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = x^2 + 2$$

$$
\begin{aligned}
y &= \sin(x) + x^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos(x) + 2x \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\sin(x) + 2 \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y &= x^2 + 2 \\
(-\sin(x) + 2) + (\sin(x) + x^2) &= x^2 + 2 \\
x^2 + 2 &= x^2 + 2
\end{aligned}
$$

Exercise 6

Determine whether the given function is a solution to the given differential equation.

$$x = \cos(t) \quad \frac{\mathrm{d}x}{\mathrm{d}t} + tx = \sin(2t)$$

$$
\begin{aligned}
x &= \cos(2t) \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= -2\sin(2t) \\
\frac{\mathrm{d}x}{\mathrm{d}t} + tx &= \sin(2t) \\
-2\sin(2t) + t(\cos(2t)) &= \sin(2t) \\
t\cos(2t) &= 3\sin(2t)
\end{aligned}
$$

No solution.

Exercise 7

Determine whether the given function is a solution to the given differential equation.

$$y = \mathrm{e}^{2x} - 3\mathrm{e}^{-x} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$$

$$
\begin{aligned}
y &= \mathrm{e}^{2x} - 3\mathrm{e}^{-x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\mathrm{e}^{2x} + 3\mathrm{e}^{-x} \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 4\mathrm{e}^{2x} - 3\mathrm{e}^{-x} \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y &= 0 \\
(4\mathrm{e}^{2x} - 3\mathrm{e}^{-x}) - (2\mathrm{e}^{2x} + 3\mathrm{e}^{-x}) - 2(\mathrm{e}^{2x} - 3\mathrm{e}^{-x}) &= 0 \\
0 &= 0
\end{aligned}
$$

Exercise 8

Determine whether the given function is a solution to the given differential equation.

$$y = 3\sin(2x) + \mathrm{e}^{-x} \quad y'' + 4y = 5\mathrm{e}^{-x}$$

$$
\begin{aligned}
y &= 3\sin(2x) + \mathrm{e}^{-x} \\
y' &= 6\cos(2x) - \mathrm{e}^{-x} \\
y'' &= -12\sin(2x) + \mathrm{e}^{-x} \\
y'' + 4y &= 5\mathrm{e}^{-x} \\
(-12\sin(2x) + \mathrm{e}^{-x}) + 4(3\sin(2x) + \mathrm{e}^{-x}) &= 5\mathrm{e}^{-x} \\
5\mathrm{e}^{-x} &= 5\mathrm{e}^{-x}
\end{aligned}
$$

Exercise 9

Determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

$$x^2 + y^2 = 4 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$$

$$
\begin{aligned}
x^2 + y^2 &= 4 \\
2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x}2y &= -2x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-x}{y}
\end{aligned}
$$

Not a solution.

Exercise 10

Determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

$$y - \ln(y) = x^2 + 1 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{y-1}$$

$$
\begin{aligned}
y - \ln(y) &= x^2 - 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{y} &= 2x \\
\frac{\mathrm{d}y}{\mathrm{d}x}(1 - \frac{1}{y}) &= 2x \\
\frac{\mathrm{d}y}{\mathrm{d}x}(y-1) &= 2xy \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2xy}{y-1}
\end{aligned}
$$

Exercise 11

Determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

$$\mathrm{e}^{xy} + y = x - 1 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^{-xy} - y}{\mathrm{e}^{-xy} + x}$$

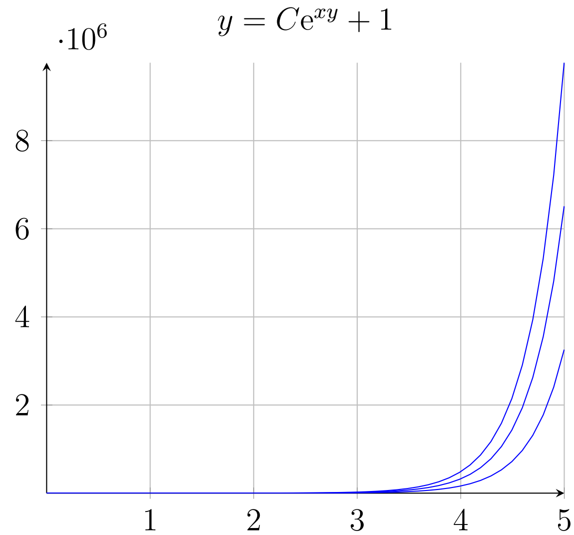
$$
\begin{aligned}
e^{xy} + y &= x - 1 \\
e^{xy}\frac{d}{dx}(xy) + \frac{dy}{dx} &= 1 \\
e^{xy}(x\frac{dy}{dx} + y) + \frac{dy}{dx} &= 1 \\
\frac{dy}{dx}xe^{xy} + ye^{xy} + \frac{dy}{dx} &= 1 \\
\frac{dy}{dx}(1 + xe^{xy}) &= 1 - ye^{xy} \\
\frac{dy}{dx} &= \frac{1 - ye^{xy}}{1 + xe^{xy}}
\end{aligned}
$$

Not a solution.

Exercise 17

Show that $\phi(x) = Ce^{3x} + 1$ is a solution to $\frac{dy}{dx} - 3y = -3$ for any choice of the constant C. Thus, $Ce^{3x} + 1$ is a one-parameter family of solutions to the differential equation. Graph several of the solution curves using the same coordinate axes.

$$
\begin{aligned}
\phi(x) &= Ce^{3x} + 1 \\
\phi'(x) &= 3Ce^{3x} \\
\frac{dy}{dx} - 3y &= -3 \\
3Ce^{3x} - 3(Ce^{3x} + 1) &= -3 \\
-3 &= -3
\end{aligned}
$$



Exercise 19

Show that the equation $(\frac{\mathrm{d}y}{\mathrm{d}x})^2 + y^2 + 4 = 0$ has no (real-valued) solution.

$$\begin{aligned}(\frac{\mathrm{d}y}{\mathrm{d}x})^2 + y^2 + 4 &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\sqrt{-y^2 - 4}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\sqrt{-1}\sqrt{y^2 + 4}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\sqrt{y^2 + 4}i\end{aligned}$$

Solutions are imaginary.

Exercise 22

Verify that the function $\phi(x) = c_1\mathrm{e}^x + c_2\mathrm{e}^{-2x}$ is a solution to the linear equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$$

for any choice of the constants c_1 and c_2.

$$\begin{aligned}\phi(x) &= c_1\mathrm{e}^x + c_2\mathrm{e}^{-2x}\\ \phi'(x) &= c_1\mathrm{e}^x - 2c_2\mathrm{e}^{-2x}\\ \phi''(x) &= c_1\mathrm{e}^x + 4c_2\mathrm{e}^{-2x}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y &= 0\\ (c_1\mathrm{e}^x + 4c_2\mathrm{e}^{-2x}) + (c_1\mathrm{e}^x - 2c_2\mathrm{e}^{-2x}) - 2(c_1\mathrm{e}^x + c_2\mathrm{e}^{-2x}) &= 0\\ 0 &= 0\end{aligned}$$

Determine c_1 and c_2 so that each of the following initial conditions is satisfied.

$$y(0) = 2 \quad y'(0) = 1$$

$$\begin{aligned}y(0) = c_1\mathrm{e}^0 + c_2\mathrm{e}^0 &= 2\\ c_1 + c_2 &= 2\\ y'(0) = c_1\mathrm{e}^0 - 2c_2\mathrm{e}^0 &= 1\\ c_1 - 2c_2 &= 1\\ (2 - c_2) - 2c_2 &= 1\\ c_2 = \frac{1}{3} \quad c_1 &= \frac{5}{3}\end{aligned}$$

$$y(1) = 1 \quad y'(1) = 0$$

$$\begin{aligned}y(1) = c_1\mathrm{e} + c_2\mathrm{e}^{-2} &= 1\\ y'(1) = c_1\mathrm{e} - 2c_2\mathrm{e}^{-2} &= 0\\ 3c_2\mathrm{e}^{-2} &= 1\\ c_2 = \frac{\mathrm{e}^2}{3} \quad c_1 &= \frac{2}{3\mathrm{e}}\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech