Differential Equations

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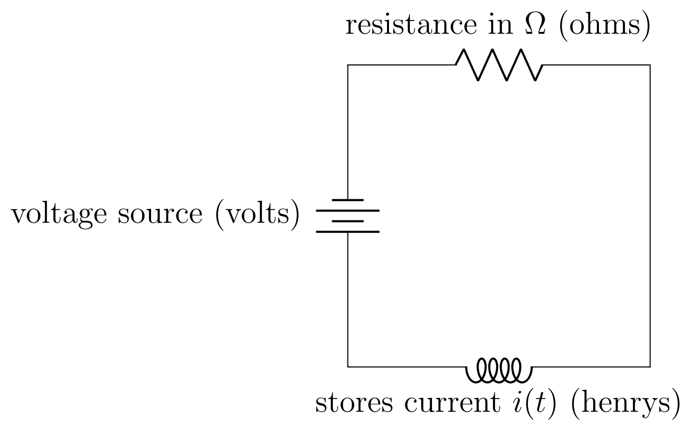
Circuits

Consider an RL series circuit:



We want to find the current at time t. The current i is in amperes. According to **Kirchhoff's Law**, the sum of the the voltage drops across the inductor and the resistor is equal to the impressed voltage $E(t)$. The governing equation is:

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = E(t)$$

where $E_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$ is the voltage drop across the inductor and Ri is the voltage drop across the resistor. This is a first order linear equation.

Example

A 12-volt battery is connected to a series circuit where the inductance is $\frac{1}{2}$H and the resistance is 10Ω. Find the current at time t if the initial current is 0. Find the steady state current.

$$
\begin{aligned}
L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri &= E(t) \\
\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}i &= \frac{E(t)}{L} \\
E &= 12V \quad L = \frac{1}{2} \quad R = 10\Omega \\
\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{10}{\frac{1}{2}}i &= \frac{12}{\frac{1}{2}} \\
\frac{\mathrm{d}i}{\mathrm{d}t} + 20i &= 24 \\
\mu(t) &= \mathrm{e}^{\int 20\ \mathrm{d}t} = \mathrm{e}^{20t} \\
\mathrm{e}^{20t}\frac{\mathrm{d}i}{\mathrm{d}t} + 20i\mathrm{e}^{20t} &= 24\mathrm{e}^{20t} \\
\int \left(\mathrm{e}^{20t}\frac{\mathrm{d}i}{\mathrm{d}t} + 20i\mathrm{e}^{20t}\right)\ \mathrm{d}t &= \int 24\mathrm{e}^{20t}\ \mathrm{d}t \\
\mathrm{e}^{20t}i &= \frac{24}{20}\mathrm{e}^{20t} + c \\
i(t) &= \frac{6}{5} + c\mathrm{e}^{-20t}
\end{aligned}
$$

Initially, $i(0) = 0$:

$$
\begin{aligned}
0 &= \frac{6}{5} + c \\
c &= -\frac{6}{5} \\
i(t) &= \frac{6}{5} - \frac{6}{5}\mathrm{e}^{-20t} \\
\lim_{t\to\infty} \frac{6}{5} - \frac{6}{5}\mathrm{e}^{-20t} &= \frac{6}{5}
\end{aligned}
$$

$\frac{6}{5}A$ is the steady state and $-\frac{6}{5}\mathrm{e}^{20t}$ is the transient term.

Example

An electromagnet is modeled as an RL circuit while it is being energized with a voltage source. If the inductance is 10H and the resistor is 30Ω, how long does it take a constant voltage to energize the electromagnet to 80% of its final value?

$$
\begin{aligned}
10\frac{\mathrm{d}i}{\mathrm{d}t} + 3i &= V \\
\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{3}{10}i &= \frac{V}{10} \\
\mu(t) &= \mathrm{e}^{\int \frac{3}{10}\ \mathrm{d}t} = \mathrm{e}^{\frac{3}{10}t} \\
\mathrm{e}^{\frac{3}{10}t}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{3}{10}\mathrm{e}^{\frac{3}{10}t}i &= \frac{1}{10}\mathrm{e}^{\frac{3}{10}t}V \\
\int \left(\mathrm{e}^{\frac{3}{10}t}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{3}{10}\mathrm{e}^{\frac{3}{10}t}i\right)\ \mathrm{d}t &= \int \frac{1}{10}\mathrm{e}^{\frac{3}{10}t}V\ \mathrm{d}t \\
\mathrm{e}^{\frac{3}{10}t}i &= \frac{1}{10}\frac{10}{3}\mathrm{e}^{\frac{3}{10}t} + c \\
i &= \frac{V}{3} + c\mathrm{e}^{-\frac{3}{10}t}
\end{aligned}
$$

Using $i(0) = 0$:

$$
\begin{aligned}
0 &= \frac{V}{3} + c \\
c &= -\frac{V}{3} \\
i(t) &= \frac{V}{3} - \frac{V}{3}\mathrm{e}^{\frac{3}{10}t} \\
0.8 &= 1 - \mathrm{e}^{\frac{-3}{10}t} \\
-0.2 &= -\mathrm{e}^{-\frac{3}{10}t} \\
\ln(0.2) &= -\frac{3}{10}t \\
t &\approx 5.4
\end{aligned}
$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech