

Differential Equations

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Separable Equations

Consider:

$$\frac{dy}{dx} = p(y)g(x)$$

In this case, we can rewrite it as:

$$\frac{dy}{p(y)} = g(x) dx$$

If we let $h(y) = \frac{1}{p(y)}$:

$$\begin{aligned}h(y) dy &= g(x) dx \\ \int h(y) dy &= \int g(x) dx \\ H(y) &= G(x) + c\end{aligned}$$

This yields an implicit solution. Separable equations can be linear or non-linear.

Example

Solve the following:

$$\frac{dy}{dx} = 8x^3 e^{-2y}$$

$$\begin{aligned} \frac{dy}{dx} &= 8x^3 e^{-2y} \\ \frac{dy}{e^{-2y}} &= 8x^3 dx \\ e^{2y} dy &= 8x^3 dx \\ \int e^{2y} dy &= \int 8x^3 dx \\ \frac{1}{2}e^{2y} + c_1 &= 2x^4 + c_2 \\ \frac{1}{2}e^{2y} &= 2x^4 + c \end{aligned}$$

We can also solve this explicitly:

$$\begin{aligned} \frac{1}{2}e^{2y} &= 2x^4 + c \\ \ln(e^{2y}) &= \ln(4x^4 + c) \\ 2y &= \ln(4x^4 + c) \\ y &= \frac{\ln(4x^4 + c)}{2} \end{aligned}$$

Example

Solve the following initial value problem:

$$\frac{dy}{dx} = (1 + y^2) \tan(x) \quad y(0) = \sqrt{3}$$

This is a non-linear separable equation.

$$\begin{aligned} \frac{dy}{dx} &= (1 + y^2) \tan(x) \\ \frac{dy}{1 + y^2} &= \tan(x) dx \\ \int \frac{dy}{1 + y^2} &= \int \tan(x) dx \\ \tan^{-1}(y) &= \ln |\sec(x)| + c \end{aligned}$$

Using our initial value $y(0) = \sqrt{3}$:

$$\tan^{-1}(\sqrt{3}) = \ln |\sec(0)| + c$$

$$\tan^{-1}(\sqrt{3}) = \ln |1| + c$$

$$\tan^{-1}(\sqrt{3}) = c$$

$$c = \frac{\pi}{3} \text{ over } \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$$

We can now solve explicitly for y :

$$\tan^{-1}(y) = \ln |\sec(x)| + \frac{\pi}{3}$$

$$\tan(\tan^{-1}(y)) = \tan \left(\ln |\sec(x)| + \frac{\pi}{3} \right)$$

$$y = \tan \left(\ln |\sec(x)| + \frac{\pi}{3} \right)$$

Example

Solve the following initial value problem over $[0, \infty)$:

$$\sqrt{y} \, dx + (1 + x) \, dy = 0 \quad y(0) = 1$$

$$\sqrt{y} \, dx + (1 + x) \, dy = 0$$

$$(1 + x) \, dy = -\sqrt{y} \, dx$$

$$\frac{dy}{dx} = (-\sqrt{y}) \frac{1}{1 + x}$$

$$\frac{dy}{\sqrt{y}} = -\frac{dx}{1 + x}$$

$$\int y^{-\frac{1}{2}} \, dy = -\int \frac{dx}{1 + x}$$

$$2y^{\frac{1}{2}} = -\ln |1 + x| + c$$

Using our initial value $y(0) = 1$:

$$2 = -\ln |1| + c$$

$$c = 2$$

We can now solve explicitly for y :

$$\begin{aligned}2\sqrt{y} &= 2 - \ln|1+x| \\ \sqrt{y} &= \frac{2 - \ln|1+x|}{2} \\ y &= \left[\frac{2 - \ln|1+x|}{2} \right]^2\end{aligned}$$

Example

Solve the initial value problem:

$$x^2 \frac{dy}{dx} = y - xy \quad y(1) = 1$$

$$x^2 \frac{dy}{dx} = y - xy$$

$$x^2 \frac{dy}{dx} = y(1-x)$$

$$\frac{dy}{dx} = (y) \left(\frac{1-x}{x^2} \right)$$

$$\frac{dy}{y} = \frac{1-x}{x^2} dx$$

$$\int \frac{dy}{y} = \int \left[\frac{1}{x^2} - \frac{1}{x} \right] dx$$

$$\ln|y| = -\frac{1}{x} - \ln|x| + c$$

$$e^{\ln|y|} = e^{-\frac{1}{x} - \ln|x| + c}$$

$$|y| = e^{-\frac{1}{x}} e^{-\ln|x|} e^c$$

$$= e^{-\frac{1}{x}} \frac{1}{|x|} e^c$$

$$\text{Let : } k = \pm e^c$$

$$y = \frac{ke^{-\frac{1}{x}}}{x}$$

Using our initial value $y(1) = 1$:

$$1 = \frac{ke^{-1}}{1}$$
$$k = e$$

We can now solve explicitly for y :

$$y = \frac{ke^{-\frac{1}{x}}}{x}$$
$$y = \frac{e^{1-\frac{1}{x}}}{x}$$

Example

Solve the initial value problem:

$$\frac{1}{\theta} \frac{dy}{d\theta} = \frac{y \sin \theta}{y^2 + 1} \quad y(\pi) = 1$$

$$\frac{1}{\theta} \frac{dy}{d\theta} = \frac{y \sin \theta}{y^2 + 1}$$

$$\frac{dy}{d\theta} = (\theta \sin \theta) \left(\frac{y}{y^2 + 1} \right)$$

$$\frac{y^2 + 1}{y} dy = \theta \sin \theta d\theta$$

$$\int \frac{y^2 + 1}{y} dy = \int \theta \sin \theta d\theta$$

$$\frac{y^2}{2} + \ln |y| + c = -\theta \cos \theta - \int (-\cos \theta d\theta)$$

$$= -\theta \cos \theta + \sin \theta + c$$

Using our initial value $y(\pi) = 1$:

$$\frac{1}{2} = -\pi \cos(\pi) + \sin(\pi) + c$$

$$\frac{1}{2} = -\pi(-1) + c$$

$$c = \frac{1}{2} - \pi$$

We can now solve explicitly for y :

$$\frac{y^2}{2} + \ln |y| = -\theta \cos \theta + \sin \theta + \left(\frac{1}{2} - \pi\right)$$

First Order Linear Equations

General form:

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

Let's consider:

$$x \frac{dy}{dx} + y = x^2$$

By the product rule we can see that the left hand side is equal to $\frac{dy}{dx} [xy]$:

$$\frac{d}{dx} [xy] = x^2$$

Integrating both sides:

$$\begin{aligned} \int \frac{d}{dx} [xy] dx &= \int x^2 dx \\ xy &= \frac{x^3}{3} + c \\ y &= \frac{x^2}{3} + \frac{c}{x} \end{aligned}$$

Standard Form

Consider:

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

Write it in standard form:

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

Let $P(x) = \frac{a_0(x)}{a_1(x)}$ and $Q(x) = \frac{g(x)}{a_1(x)}$:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

This is a first order linear equation in standard form. In order to solve this, we need to determine a standard function of x called μ . We multiply the equation by μ :

$$\mu(x) \frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

We require that that $\mu'(x) = \mu(x)P(x)$. To find this function μ :

$$\begin{aligned}\frac{\mu'(x)}{\mu(x)} &= P(x) \\ \int \frac{\mu'(x)}{\mu(x)} &= \int \mu P(x) \\ \ln |\mu(x)| &= \int P(x) \, dx + c \\ c &= 0 \\ e^{\ln |\mu|} &= e^{\int P(x) \, dx} \\ |\mu(x)| &= e^{\int P(x) \, dx}\end{aligned}$$

This is called an integrating factor. From this, we can determine that:

$$\begin{aligned}\frac{d}{dx} [\mu(x)y] &= \mu(x)Q(x) \\ \int \frac{d}{dx} [\mu(x)y] &= \int \mu(x)Q(x) \\ \mu(x)y &= \int \mu(x)Q(x) \, dx \\ y &= \frac{1}{\mu(x)} \left[\int \mu(x)Q(x) \, dx \right] \\ \mu(x) &= e^{\int P(x) \, dx}\end{aligned}$$

Example

$$\frac{dy}{dx} - 3y = 6$$

This is a first order linear equation already in standard form. It is also separable, but we will solve using the method for first order linear equations.

1. Identify $P(x)$:

$$P(x) = -3 \quad Q(x) = 6$$

2. Find $\mu(x)$:

$$\mu(x) = e^{\int -3 \, dx} = e^{-3x}$$

3. Multiply the given equation by $\mu(x)$:

$$\begin{aligned}e^{-3x} \frac{dy}{dx} - 3e^{-3x}y &= 6e^{-3x} \\ \frac{d}{dx} \left[e^{-3x}y \right] &= 6e^{-3x} \\ \int \frac{d}{dx} \left[e^{-3x}y \right] dx &= \int 6e^{-3x} dx \\ e^{-3x}y &= 6\left(-\frac{1}{3}e^{-3x}\right) + c \\ &= -2e^{-3x} + c \\ y &= -2 + Ce^{3x}\end{aligned}$$

Example

$$x \frac{dy}{dx} - 4y = x^6 e^x$$

1. Write it in standard form:

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x$$

2. Identify $P(x)$:

$$P(x) = \frac{-4}{x}$$

3. Find $\mu(x)$:

$$\mu(x) = e^{\int \frac{-4}{x} dx} = e^{-4 \ln|x|} = e^{\ln \frac{1}{x^4}} = \frac{1}{x^4}$$

4. Multiply the given equation by $\mu(x)$:

$$\begin{aligned}\frac{1}{x^4} \frac{dy}{dx} - \frac{4}{x^5}y &= xe^x \\ \frac{d}{dx} \left[\frac{1}{x^4}y \right] &= xe^x \\ \frac{1}{x^4}y &= \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c \\ y &= x^5 e^x - x^4 e^x + Cx^4\end{aligned}$$

Example

Solve the following over $(0, \infty)$:

$$\begin{aligned}x \frac{dy}{dx} + 3(y + x^2) &= \frac{\cos(x)}{x^2} \\x \frac{dy}{dx} + 3y &= \frac{\cos(x)}{x^2} - 3x^2 \\ \frac{dy}{dx} + \frac{3}{x}y &= \frac{\cos(x)}{x^3} - 3x \\ P(x) &= \frac{3}{x} \\ \mu(x) &= e^{\int \frac{3}{x} dx} = e^{3 \ln(x)} = e^{\ln(x^3)} = x^3 \\ x^3 \frac{dy}{dx} + 3x^2y &= \cos(x) + 3x^4 \\ \frac{d}{dx} [x^3y] &= \cos(x) + 3x^4 \\ \int \frac{d}{dx} [x^3y] dx &= \int \cos(x) + 3x^4 dx \\ x^3y &= \sin(x) + \frac{3x^5}{5} + c \\ y &= \frac{\sin(x)}{x^3} - \frac{3x^2}{5} + \frac{c}{x^3}\end{aligned}$$

Example

Solve the following initial value problem over $(-\frac{\pi}{2}, \frac{\pi}{2})$ given $y(0) = 2$:

$$\begin{aligned}\cos(x) \frac{dy}{dx} + \sin(x)y &= 1 \\ \frac{dy}{dx} + \frac{\sin(x)}{\cos(x)}y &= \frac{1}{\cos(x)} \\ \frac{dy}{dx} + \tan(x)y &= \sec(x) \\ \mu(x) &= e^{\int P(x) dx} = e^{\int \tan(x) dx} = e^{\ln |\sec(x)|} = \sec(x) \\ \sec(x) \frac{dy}{dx} + \sec(x) \tan(x)y &= \sec^2(x)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx} [\sec(x)y] &= \sec^2(x) \\ \int \frac{d}{dx} [\sec(x)y] dx &= \int \sec^2(x) dx \\ \sec(x)y &= \tan(x) + c \\ y &= \frac{\tan(x)}{\sec(x)} + \frac{c}{\sec(x)} \\ &= \sin(x) + c \cos(x)\end{aligned}$$

Using our initial value:

$$\begin{aligned}y &= \sin(x) + c \cos(x) \\ 2 &= \sin(0) + c \cos(0) \\ &= 0 + c \\ c &= 2 \\ y &= \sin(x) + 2 \cos(x)\end{aligned}$$

Substitutions and Transformations

Consider:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where n is any real number. This is called a **Bernoulli Equation**. We use the substitution $v = y^{1-n}$ to transform the equation into a new variable:

$$\frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}$$

This gives a linear equation in v :

$$\frac{dv}{dx} = \frac{1}{1-n}y^n \frac{dv}{dx}$$

We divide the equation by y^n :

$$y^{-n} \frac{dy}{dx} + P(x)y(y^{-n}) = Q(x)$$

We can now substitute for $\frac{dy}{dx}$:

$$\begin{aligned} \frac{1}{1-n}y^n \frac{dv}{dx}y^{-n} + P(x)y^{1-n} &= Q(x) \\ \frac{1}{1-n} \frac{dv}{dx} + P(x)v &= Q(x) \\ \frac{dv}{dx} + (1-n)P(x)v &= (1-n)Q(x) \end{aligned}$$

Using this substitution allows us to change our equation into a first-order linear equation in standard form, which we can now solve.

Example

Solve the following over $(0, \infty)$:

$$\begin{aligned}\frac{dy}{dx} + \frac{y}{x} &= x^2 y^2 \\ \frac{dy}{dx} + \frac{1}{x} y &= x^2 y^2 \\ v &= y^{1-2} = y^{-1}\end{aligned}$$

$$\begin{aligned}\frac{dv}{dx} &= -y^{-2} \frac{dy}{dx} \\ \frac{dy}{dx} &= -y^2 \frac{dv}{dx}\end{aligned}$$

$$\begin{aligned}\frac{1}{y^2} \left(\frac{dy}{dx} + \frac{1}{x} y = x^2 y^2 \right) \\ y^{-2} \frac{dy}{dx} + \frac{1}{x} y(y^{-2}) &= x^2 \\ y^{-2} \left(-y^2 \frac{dv}{dx} \right) + \frac{1}{x} y^{-1} &= x^2 \\ -\frac{dv}{dx} + \frac{1}{x} v &= x^2 \\ \frac{dv}{dx} - \frac{1}{x} v &= -x^2\end{aligned}$$

$$\frac{dv}{dx} - \frac{1}{x} v = -x^2$$

$$\mu(x) = e^{\int -\frac{dx}{x}} = e^{-\ln|x|} = e^{\ln|\frac{1}{x}|} = \frac{1}{x}$$

$$\begin{aligned}\frac{1}{x} \frac{dv}{dx} - \frac{1}{x^2} v &= -x \\ \int \frac{d}{dx} \left[\frac{1}{x} v \right] dx &= - \int x dx \\ \frac{1}{x} v &= -\frac{x^2}{2} + c \\ v &= -\frac{x^3}{2} + cx \\ \frac{1}{y} &= -\frac{x^3}{2} + cx\end{aligned}$$

Example

Solve the following initial value problem given $y(1) = 1$:

$$\begin{aligned}x^2 \frac{dy}{dx} - 2xy &= 3y^4 \\ \frac{dy}{dx} - \frac{2}{x}y &= \frac{3y^4}{x^2} \\ n = 4 \quad v &= y^{1-4} = y^{-3} \\ \frac{dv}{dx} &= -3y^{-4} \frac{dy}{dx} \\ \frac{dy}{dx} &= -\frac{1}{3}y^4 \frac{dv}{dx} \\ \frac{1}{y^4} \left(\frac{dy}{dx} \right) &= -\frac{1}{3}y^4 \frac{dv}{dx} \\ y^{-4} \frac{dy}{dx} - \frac{2}{x}y(y^{-4}) &= \frac{3}{x^2} \\ y^{-4} \left(-\frac{1}{3}y^4 \frac{dv}{dx} \right) - \frac{2}{x}v &= \frac{3}{x^2} \\ -\frac{1}{3} \frac{dv}{dx} - \frac{2}{x}v &= \frac{3}{x^2} \\ \frac{dv}{dx} + \frac{6}{x}v &= -\frac{9}{x^2}\end{aligned}$$

This equation is now a first order linear equation in v .

$$\begin{aligned}\frac{dv}{dx} + \frac{6}{x}v &= -\frac{9}{x^2} \\ \mu(x) &= e^{\int \frac{6}{x} dx} = e^{6 \ln|x|} = e^{\ln|x^6|} = x^6 \\ x^6 \frac{dv}{dx} + \frac{6x^5}{v} &= -9x^4 \\ \frac{d}{dx} [x^6 v] &= -9x^4 \\ \int \frac{d}{dx} [x^6 v] dx &= \int -9x^4 dx \\ x^6 v &= -\frac{9}{5}x^5 + c \\ v &= -\frac{9}{5x} + \frac{c}{x^6} \\ \frac{1}{y^3} &= -\frac{9}{5x} + \frac{c}{x^6}\end{aligned}$$

Using our initial value:

$$\begin{aligned}\frac{1}{y^3} &= -\frac{9}{5x} + \frac{c}{x^6} \\ \frac{1}{1^3} &= -\frac{9}{5} + \frac{c}{1^6} \\ c &= \frac{14}{5} \\ \frac{1}{y^3} &= -\frac{9}{5x} + \frac{14}{5} \frac{1}{x^6}\end{aligned}$$

Exact Equations

Consider:

$$z = f(x, y) \quad \text{vs} \quad y = f(x)$$

$z = f(x, y)$ is a surface in 3D, while $y = f(x)$ is a curve in 2D. Recall that when differentiating $y = f(x)$, $\frac{dy}{dx} = f'(x)$ and $dy = f'(x) dx$. To differentiate $z = f(x, y)$ we need to find partial derivatives. These are computed by treating y or x as a constant and taking the derivative with respect to other variable.

$$\begin{aligned}x &= f(x, y) = x^3y^4 - 2x^3y^5 \\f_x &= \frac{\partial f}{\partial x} = 3x^2y^4 - 6x^2y^5 \\f_y &= \frac{\partial f}{\partial y} = 4x^3y^3 - 10x^3y^4 \\f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = 12x^2y^3 - 30x^2y^4 \\f_{yx} &= \frac{\partial^2 f}{\partial y \partial x} = 12x^2y^3 - 30x^2y^4\end{aligned}$$

We can also integrate $f(x, y)$ with respect to x or with respect to y .

$$\int xy^3 dy = x \int y^2 dy = \frac{xy^3}{3} + c$$

dy tells us the variable of integration, with x being constant with respect to y .

$$\int xy^2 dx = \frac{x^2y^2}{2} + c$$

dx tells us the variable of integration, with y being constant with respect to x . Consider again $z = f(x, y)$, the total differential is:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

This is a first order equation. This equation is called an exact equation if:

$$\frac{\partial}{\partial y} \left[\frac{\partial f}{\partial x} \right] = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right]$$

We can write this as:

$$M(x, y) dx + N(x, y) dy = 0$$

where $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$. We want to find $f(x, y) = c$, geometrically represented as a level curve, or a slice of the curve $z = f(x, y)$. Initially:

$$f(x, y) = \int M(x, y) dx + h(y)$$

We need to find $h(y)$ by differentiating with respect to y :

$$\begin{aligned}\frac{\partial}{\partial y} \left[\int M(x, y) dy \right] &= h'(y) \\ \int h'(y) dy &= h(y) + c \\ f(x, y) &= \int M(x, y) dx + h(y) + c = 0\end{aligned}$$

Example

Solve the initial value problem given $y(1) = 1$:

$$\begin{aligned}(x^2y^3) dx + (x^3y^2) dy &= 0 \\ M(x, y) &= x^2y^3 \quad N(x, y) = x^3y^2 \\ \frac{\partial M}{\partial y} &= 3x^2y^2 \quad \frac{\partial N}{\partial x} = 3x^2y^2\end{aligned}$$

From this, we can determine that it is an exact equation.

$$\begin{aligned}f(x, y) &= \int x^2y^3 dx + h(y) \\ &= \frac{x^3y^3}{3} + h(y) \\ \frac{\partial f}{\partial y} &= x^3y^2 + h'(y) = N(x, y) = x^3y^2 \\ \therefore h'(y) &= 0 \\ h(y) &= k \\ f(x, y) &= c = \frac{x^3y^3}{3} + k \\ \frac{x^3y^3}{3} &= c\end{aligned}$$

Using our initial value:

$$\begin{aligned}\frac{x^3 y^3}{3} &= c \\ \frac{(1)(1)}{3} &= c \\ \frac{x^3 y^3}{3} &= \frac{1}{3}\end{aligned}$$

Example

$$(e^{2y} - y \cos(xy)) dx + (2xe^{2y} - x \cos(xy) + 2y) dy = 0$$

$$M(x, y) = e^{2y} - y \cos(xy) \quad N(x, y) = 2xe^{2y} - x \cos(xy) + 2y$$

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2e^{2y} - (\cos(xy) + y(-\sin(xy)x)) \\ &= 2e^{2y} - \cos(xy) + xy \sin(xy)\end{aligned}$$

$$\begin{aligned}\frac{\partial N}{\partial x} &= 2e^{2y} - (\cos(xy) + x(-\sin(xy))y) \\ &= 2e^{2y} - \cos(xy) + xy \sin(xy)\end{aligned}$$

$$f(x, y) = \int M(x, y) + h(y)$$

$$= \int (e^{2y} - y \cos(xy)) dx + h(y)$$

$$= xe^{2y} - (y + \sin(xy))\frac{1}{y} + h(y)$$

$$= xe^{2y} - \sin(xy) + h(y)$$

$$\frac{\partial f}{\partial y} = 2xe^{2y} - \cos(xy)x + h'(y) = N(x, y)$$

$$h'(y) = 2y$$

$$h(y) = y^2 + c$$

$$\therefore f(x, y) = xe^{2y} - \sin(xy) + y^2 + c$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech