

Differential Equations

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Concept Review

$$\int \frac{dx}{1+x^2} \neq \ln(1+x^2) + c$$
$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$$

Example

$$\int \frac{x}{1+x^2} dx$$

Let : $u = x^2 + 1$
 $du = 2x dx$

$$\frac{du}{2} = x dx$$
$$\int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{du}{u}$$
$$= \frac{1}{2} \ln |u| + c$$
$$= \frac{1}{2} \ln |x^2 + 1| + c$$

Example

$$\begin{aligned}\int e^{2x} dx &\neq \frac{e^{2x+1}}{2x+1} + c \\ &= \frac{1}{2}e^{2x} + c\end{aligned}$$

Example

$$\begin{aligned}\int \ln(x) dx \\ \text{Let : } u = \ln(x) \quad dv = dx \\ du = \frac{1}{x} dx \quad v = x \\ = x \ln(x) - \int x \frac{1}{x} dx\end{aligned}$$

Properties of Logs

1. $\ln(ab) = \ln(a) + \ln(b)$ given $a, b \neq 0$
2. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
3. $r \ln(a) = \ln(a^r)$

Rules of Exponents

1. $e^{x+y} = e^x e^y$
2. $e^{x-y} = \frac{e^x}{e^y}$
3. $e^{x^y} = e^{xy}$

Solutions and Initial Value Problems

A differential equation is an equation that contains one or more derivatives of some unknown function.

$$y'' - \frac{2}{x^2}y = 0$$

Using Leibniz's Notation:

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y(x) = 0$$

Another example:

$$y'' + 3y' + 2y = 0$$

A function ϕ defined on some interval I having at least n continuous derivatives on I , is an explicit solution over I if it satisfies the equation on I .

Classification by Order and Linearity

The order of a differential equation is the order of the highest derivative that appears in the equation.

$$y''' + 2y' + y = e^x$$

is a third order equation. With respect to linearity, consider the following:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where $a_n(x)$ is a function of the independent variable only, $y^{(n)}$ is a derivative to the n^{th} power and $g(x)$ is a function of x only.

- $\frac{d^2y}{dx^2} = -\cos(y)$ is a non-linear, second-order differential equation.
- $y'' + \ln(y)y' - 5y = e^x$ is a non-linear, second-order differential equation.
- $y'' + \ln(x)y - 5y = e^x$ is a linear, second-order differential equation.

Example

Show that $y = \phi(x) = x^2 - \frac{1}{x}$ is an explicit solution to $y'' - \frac{2}{x^2}y = 0$.

$$\begin{aligned}y &= x^2 - \frac{1}{x} \\y' &= 2x + x^{-2} \\y'' &= 2 - 2x^{-3} \\y'' - 2\frac{x^2}{y} &= 2 - 2x^{-3} - \frac{2}{x^2}\left(x^2 - \frac{1}{x}\right) \\&= 2 - \frac{2}{x^3} - 2 + \frac{2}{x^3} \\&= 0\end{aligned}$$

Hence, y satisfies this differential equation.

Example

Verify that $y(t) = e^{-2t} \sin(4t)$ is a solution to $y'' + 4y' + 20y = 0$.

$$\begin{aligned}y &= e^{-2t} \sin(4t) \\y' &= -2e^{-2t} \sin(4t) + e^{-2t}(4) \cos(4t) \\y'' &= 4e^{-2t} \sin(4t) + (-2e^{-2t})(4) \cos(4t) + (4)(-2e^{-2t}) \cos(4t) + e^{-2t}(-16) \sin(4t) \\y'' + 4y' + 20y &= 4e^{-2t} \sin(4t) - 8e^{-2t} \cos(4t) - 8e^{-2t} \cos(4t) - \\&\quad 16e^{-2t} \sin(4t) + 4\left(-2e^{-2t} \sin(4t) + 4e^{-2t} \cos(4t)\right) + \\&\quad 20e^{-2t} \sin(4t) \\&= 24e^{-2t} \sin(4t) - 24e^{-2t} \sin(4t) + 16e^{-2t} \cos(4t) - 16e^{-2t} \cos(4t) \\&= 0\end{aligned}$$

Example

Verify that $y = c_1e^t + c_2e^{2t}$ is an explicit solution to $y'' - 3y' + 2y = 0$ for any constants c_1 and c_2 .

$$\begin{aligned}y &= c_1e^t + c_2e^{2t} \\y' &= c_1e^t + 2c_2e^{2t} \\y'' &= c_1e^t + 4c_2e^{2t} \\y'' - 3y' + 2y &= c_1e^t + 4c_2e^{2t} - 3(c_1e^t + 2c_2e^{2t}) + 2(c_1e^t + c_2e^{2t}) \\&= 0\end{aligned}$$

Implicit Solutions

A relation $G(x, y) = 0$, is an **implicit** solution if it determines one or more explicit solutions. For example, verify that $x^2 - \sin(x + y) = 1$ is an implicit solution to $\frac{dy}{dx} = 2x \sec(x + y) - 1$. Differentiating $x^2 - \sin(x + y) = 1$ implicitly, we get:

$$\begin{aligned}2x - \cos(x + y) \frac{d}{dx}(x + y) &= 0 \\2x - \cos(x + y) \left(1 + \frac{dy}{dx}\right) &= 0 \\2x - \cos(x + y) - \cos(x + y) \frac{dy}{dx} &= 0 \\2x - \cos(x + y) &= \cos(x + y) \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{2x}{\cos(x + y)} - 1 \\ &= 2x \sec(x + y) - 1\end{aligned}$$

Example

Show $x + y + e^{xy} = 0$ is an implicit solution to $(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0$.

$$\begin{aligned}x + y + e^{xy} &= 0 \\1 + \frac{dy}{dx} + e^{xy} \left[\frac{d}{dx}(xy) \right] &= 0 \\1 + \frac{dy}{dx} + e^{xy} \left[y + x \frac{dy}{dx} \right] &= 0 \\1 + \frac{dy}{dx} + ye^{xy} + xe^{xy} \frac{dy}{dx} &= 0 \\1 + \left[1 + xe^{xy} \right] \frac{dy}{dx} + ye^{xy} &= 0 \\(1 + xe^{xy}) \frac{dy}{dx} + 1 + ye^{xy} &= 0\end{aligned}$$

Example

Show that $x^2 + y^2 - 25 = 0$ is an implicit solution to $\frac{dy}{dx} = f(x, y) = \frac{-x}{y}$ on $(-5, 5)$.

$$\begin{aligned}x^2 + y^2 - 25 &= 0 \\2x + 2y \frac{dy}{dx} &= 0 \\y \frac{dy}{dx} &= -x \\ \frac{dy}{dx} &= \frac{-x}{y}\end{aligned}$$

Alternatively, we can try doing this explicitly by solving for y :

$$\begin{aligned}y^2 &= 25 - x^2 \\y &= \pm\sqrt{25 - x^2} \\ \text{Choose } y &= \sqrt{25 - x^2} \\ &= (25 - x^2)^{\frac{1}{2}} \\ y' &= \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (-2x) \\ &= \frac{-x}{\sqrt{25 - x^2}} \\ &= \frac{-x}{y}\end{aligned}$$

We want to find where $25 - x^2 > 0$, which tells us that $-5 < x < 5$.

Families of Solutions

Let's try to solve the following:

$$\begin{aligned}\frac{dy}{dx} &= \frac{-y}{x} = f(x, y) \\ &= (-y)\frac{1}{x} \\ \frac{dy}{y} &= -\frac{dx}{x} \\ \int \frac{dy}{y} &= -\int \frac{dx}{x} \\ \ln |y| &= -\ln |x| + c\end{aligned}$$

This is an **implicit** solution to our differential equation. We can also try to solve it explicitly by exponentiating both sides.

$$\begin{aligned}\ln |y| &= -\ln |x| + c \\ e^{\ln |y|} &= e^{(-\ln |x| + c)} \\ |y| &= e^{-\ln |x|} e^c \\ |y| &= e^{\ln |x^{-1}|} e^c \\ &= \left| \frac{1}{x} \right| e^c\end{aligned}$$

If we let $k = \pm e^c$, then:

$$\begin{aligned}y &= \pm \frac{1}{x} e^c \\ &= \frac{k}{x}\end{aligned}$$

This is a **family** of one parameter k solutions. It is a general solution that contains an arbitrary constant k . Suppose we are given an initial value $y(1) = 5$, which lies on one of the curves.

$$y = \frac{k}{x} \implies 5 = \frac{k}{1} \implies k = 5$$

$y = \frac{5}{x}$ is a **particular** solution since there are no arbitrary constants.

n^{th} Order Initial Value Problems

Given $F(x, y, y', y'', \dots, y^{(n)}) = 0$ and n initial conditions:

$$\begin{aligned}y(x_0) &= y_0 \\y'(x_0) &= y_1 \\y''(x_0) &= y_2 \\&\vdots \\y^{(n-1)}(x_0) &= y_{n-1}\end{aligned}$$

We have an initial value problem. The solution over some interval I is called a particular solution.

Example

Verify that $y = c_1 \cos(x) + c_2 \sin(x) - \sin(2x)$ is a general solution to $y'' + y = 3 \sin(2x)$.

$$\begin{aligned}y &= c_1 \cos(x) + c_2 \sin(x) - \sin(2x) \\y' &= -c_1 \sin(x) + c_2 \cos(x) - 2 \cos(2x) \\y'' &= -c_1 \cos(x) - c_2 \sin(x) + 4 \sin(2x) \\y'' + y &= 3 \sin(2x)\end{aligned}$$

$$\begin{aligned}-c_1 \cos(x) - c_2 \sin(x) + 4 \sin(2x) + c_1 \cos(x) + c_2 \sin(x) - \sin(2x) &= 3 \sin(2x) \\3 \sin(2x) &= 3 \sin(2x)\end{aligned}$$

Find constants c_1 and c_2 so that the initial conditions $y(0) = 1$ and $y'(0) = 0$ are satisfied.

$$\begin{aligned}y &= c_1 \cos(x) + c_2 \sin(x) - \sin(2x) \\y(0) = 1 &= c_1 \cos(0) \\c_1 &= 1 \\y' &= -c_1 \sin(x) + c_2 \cos(x) - 2 \cos(2x) \\y'(0) = 0 &= 0 + c_2 - 2 \\c_2 &= 2 \\y &= \cos(x) + 2 \sin(x) - \sin(2x)\end{aligned}$$

Example

Determine whether $y = c_1e^{2t} + c_2te^{2t}$ is a general solution to $y'' - 4y' + 4y = 0$.

$$\begin{aligned}y &= c_1e^{2t} + c_2te^{2t} \\y' &= 2c_1e^{2t} + c_2e^{2t} + 2c_2te^{2t} \\y'' &= 4c_1e^{2t} + 2c_2e^{2t} + 2c_2e^{2t} + 4c_2te^{2t} \\y'' + 4y' + 2y &= 4c_1e^{2t} + 2c_2e^{2t} + 2c_2e^{2t} + 4c_2te^{2t} - \\&\quad 4(2c_1e^{2t} + c_2e^{2t} + 2c_2te^{2t}) + \\&\quad 4(c_1e^{2t} + c_2te^{2t}) \\&= 0\end{aligned}$$

Find constants c_1 and c_2 such that the initial conditions $y(0) = 1$ and $y'(0) = 0$ are satisfied.

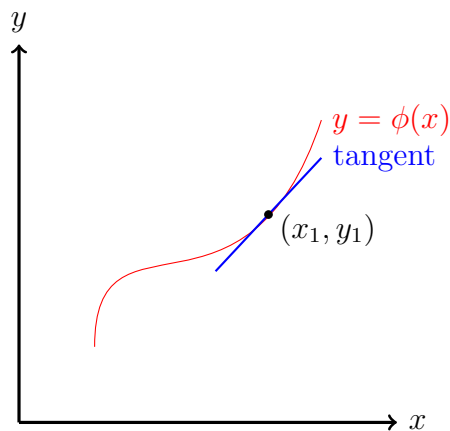
$$\begin{aligned}y(0) &= 1 = c_1 \\y'(0) &= 0 = 2 + c_2 \\c_2 &= -2\end{aligned}$$

Direction Fields of 1st Order Equations

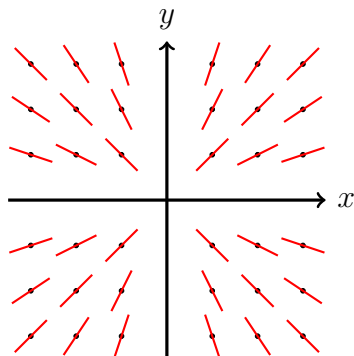
Consider the following:

$$\frac{dy}{dx} = f(x, y)$$

x is the independent variable and y is the dependent variable.



The idea is that the actual solution curve is $y = \phi(x)$. If we evaluate $\frac{dy}{dx} = f(x_1, y_1)$, it gives the slope of the tangent at (x_1, y_1) . We can construct a direction field with this. For example, if we have the differential equation $\frac{dy}{dx} = \frac{-y}{x}$.



x	y	$\frac{dy}{dx}$
1	1	-1
1	2	-2
1	3	-3
1	-1	1
-1	1	1

Autonomous First-Order Equations

Autonomous first-order equations are equations of the form

$$\frac{dy}{dx} = f(y)$$

where the independent variable x does not appear in the equation. Consider the following:

$$f(y) = \frac{dy}{dx} = y - y^2$$

This is an autonomous equation. In contrast:

$$\frac{dy}{dx} = xy = f(x, y)$$

This is not an autonomous equation. Equilibrium solutions are found by solving $f(y) = 0 = \frac{dy}{dx}$.

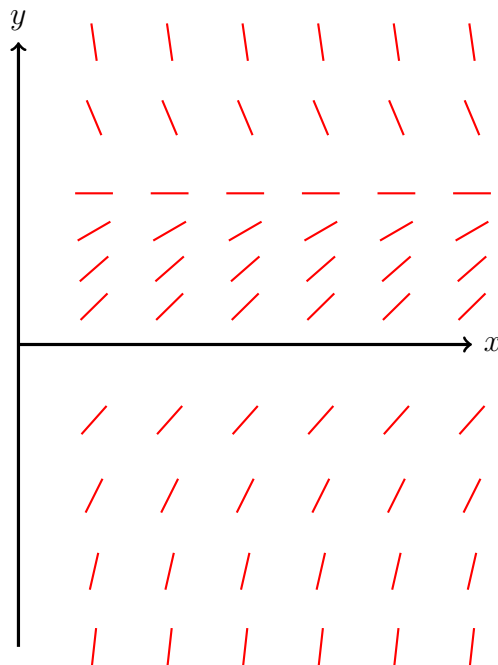
- If y is bounded by a critical point, then the graph of y approaches $y(t) = c$ as $t \rightarrow +\infty$.
- $\frac{dy}{dt}$ is either positive or negative except where $\frac{dy}{dt} = 0$.
- y is monotonic, either increasing or decreasing, there is no oscillation.
- The graph of non-constant solutions never cross an equilibrium solution.

Example

A model for the velocity v at time t of an object falling under the influence of gravity in a viscous medium is given by

$$\frac{dv}{dt} = 1 - \frac{v^3}{8}$$

Construct a direction field and determine the “terminal” velocity. Find values where $1 - \frac{v^3}{8} = 0$.



$$1 - \frac{v^3}{8} = 0 \Rightarrow v = 2$$

From this diagram, we can see all the families of solutions for this differential equation, but they all converge to $v = 2$ as $t \rightarrow \infty$.

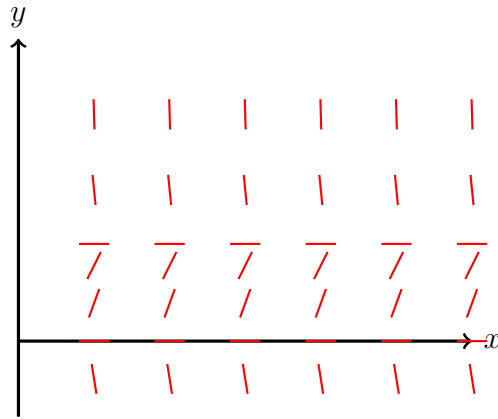
Example

Consider the following equation for a population (in thousands) of a certain species (Logistic Equation).

$$\frac{dp}{dt} = 9p - 7p^2 = f(p)$$

$$\frac{dp}{dt} = 9p - 7p^2 = 0 \Rightarrow p(9 - 7p) = 0$$

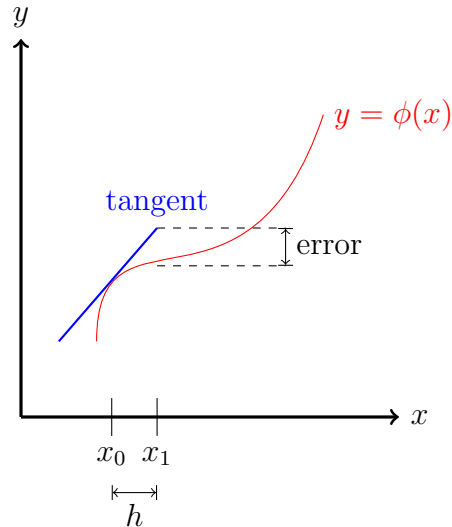
Our solutions are $p = 0$ or $p = \frac{9}{7}$.



$$\lim_{t \rightarrow \infty} p = \frac{9}{7}$$

Euler's Method

Euler's method is also known as the tangent line method for finding numerical approximations to first-order initial value problems.



With this method, we also begin with some solution curve $y = \phi(x)$ and some given (x_0, y_0) that lies on the solution curve.

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

$f(x_0, y_0)$ is the slope of the tangent to the curve at (x_0, y_0) . Given (x_0, y_0) , a single step h , and the slope $f(x_0, y_0)$:

$$y - y_0 = h[f(x_0, y_0)]$$

$$y_1 = y_0 + h[f(x_0, y_0)]$$

This is an iterative process defined by:

$$y_{n+1} = y_n + h[f(x_n, y_n)]$$

We determine the step size and iterate:

x_0	y_0
$x_1 = x_0 + h$	y_1
$x_2 = x_1 + h$	y_2
\vdots	\vdots
$x_{n+1} = x_n + h$	y_{n+1}

Example

$$\frac{dy}{dx} = x^2 + y^2 = f(x, y)$$

If we take $y(1) = 1$ with $h = 0.1$ as our step size:

$x_0 = 1$	$y_0 = 1$
$x_1 = 1.1$	$y_1 = 1 + 0.1(2) = 1.2$
$x_2 = 1.2$	$y_2 = 1.2 + 0.1[(1.1)^2 + (1.2)^2] = 1.465$
\vdots	\vdots

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech