Differential Equations

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January 2018 - May 2018

Concept Review

$$\int \frac{\mathrm{d}x}{1+x^2} \neq \ln(1+x^2) + c$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \tan^{-1}(x) + c$$

Example

$$\begin{aligned}
&\int \frac{x}{1+x^2}\ \mathrm{d}x \\
&\quad Let: u = x^2 + 1 \\
&\quad \mathrm{d}u = 2x\ \mathrm{d}x \\
&\frac{\mathrm{d}u}{2} = x\ \mathrm{d}x \\
&\int \frac{x}{1+x^2}\ \mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}u}{u} \\
&\qquad = \frac{1}{2}\ln|u| + c \\
&\qquad = \frac{1}{2}\ln|x^2+1| + c
\end{aligned}$$

Example

$$\begin{aligned}\int \mathrm{e}^{2x}\ \mathrm{d}x &\neq \frac{\mathrm{e}^{2x+1}}{2x+1} + c \\ &= \frac{1}{2}\mathrm{e}^{2x} + c\end{aligned}$$

Example

$$\begin{aligned}&\int \ln(x)\ \mathrm{d}x \\ &\quad Let: u = \ln(x) \qquad \mathrm{d}v = \ \mathrm{d}x \\ &\qquad \mathrm{d}u = \ \mathrm{d}x \quad v = x \\ &\qquad\quad = x\ln(x) - \int x\ \mathrm{d}x\end{aligned}$$

Properties of Logs

1. $\ln(ab) = \ln(a) + \ln(b)$ given $a, b \neq 0$
2. $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$
3. $r\ln(a) = \ln(a^r)$

Rules of Exponents

1. $\mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y$
2. $\mathrm{e}^{x-y} = \frac{\mathrm{e}^x}{\mathrm{e}^y}$
3. $\mathrm{e}^{x^y} = \mathrm{e}^{xy}$

Solutions and Initial Value Problems

A differential equation is an equation that contains one or more derivatives of some unknown function.

$$y'' - \frac{2}{x^2}y = 0$$

Using Leibniz's Notation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{2}{x^2}y(x) = 0$$

Another example:

$$y'' + 3y' + 2y = 0$$

A function ϕ defined on some interval I having at least n continuous derivatives on I, is an explicit solution over I if it satisfies the equation on I.

Classification by Order and Linearity

The order of a differential equation is the order of the highest derivative that appears in the equation.

$$y''' + 2y' + y = \mathrm{e}^x$$

is a third order equation. With respect to linearity, consider the following:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where $a_n(x)$ is a function of the independent variable only, $y^{(n)}$ is a derivative to the n^{th} power and $g(x)$ is a function of x only.

- $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\cos(y)$ is a non-linear, second-order differential equation.
- $y'' + \ln(y)y' - 5y = \mathrm{e}^x$ is a non-linear, second-order differential equation.
- $y'' + \ln(x)y - 5y = \mathrm{e}^x$ is a linear, second-order differential equation.

Example

Show that $y = \phi(x) = x^2 - \frac{1}{x}$ is an explicit solution to $y'' - \frac{2}{x^2}y = 0$.

$$\begin{aligned}
y &= x^2 - \frac{1}{x} \\
y' &= 2x + x^{-2} \\
y'' &= 2 - 2x^{-3} \\
y'' - 2\frac{x^2}{y} &= 2 - 2x^{-3} - \frac{2}{x^2}(x^2 - \frac{1}{x}) \\
&= 2 - \frac{2}{x^3} - 2 + \frac{2}{x^3} \\
&= 0
\end{aligned}$$

Hence, y satisfies this differential equation.

Example

Verify that $y(t) = \mathrm{e}^{-2t}\sin(4t)$ is a solution to $y'' + 4y' + 20y = 0$.

$$\begin{aligned}
y &= \mathrm{e}^{-2t}\sin(4t) \\
y' &= -2\mathrm{e}^{-2t}\sin(4t) + \mathrm{e}^{-2t}(4)\cos(4t) \\
y'' &= 4\mathrm{e}^{-2t}\sin(4t) + (-2\mathrm{e}^{-2t})(4)\cos(4t) + (4)(-2\mathrm{e}^{-2t})\cos(4t) + \mathrm{e}^{-2t}(-16)\sin(4t) \\
y'' + 4y' + 20y &= 4\mathrm{e}^{-2t}\sin(4t) - 8\mathrm{e}^{-2t}\cos(4t) - 8\mathrm{e}^{-2t}\cos(4t) - \\
&\quad 16\mathrm{e}^{-2t}\sin(4t) + 4\Big(-2\mathrm{e}^{-2t}\sin(4t) + 4\mathrm{e}^{-2t}\cos(4t)\Big) + \\
&\quad 20\mathrm{e}^{-2t}\sin(4t) \\
&= 24\mathrm{e}^{-2t}\sin(4t) - 24\mathrm{e}^{-2t}\sin(4t) + 16\mathrm{e}^{-2t}\cos(4t) - 16\mathrm{e}^{-2t}\cos(4t) \\
&= 0
\end{aligned}$$

Example

Verify that $y = c_1\mathrm{e}^t + c_2\mathrm{e}^{2t}$ is an explicit solution to $y'' - 3y' + 2y = 0$ for any constants c_1 and c_2.

$$\begin{aligned}
y &= c_1\mathrm{e}^t + c_2\mathrm{e}^{2t} \\
y' &= c_1\mathrm{e}^t + 2c_2\mathrm{e}^{2t} \\
y'' &= c_1\mathrm{e}^t + 4c_2\mathrm{e}^{2t} \\
y'' - 3y' + 2y &= c_1\mathrm{e}^t + 4c_2\mathrm{e}^{2t} - 3\left(c_1\mathrm{e}^t + 2c_2\mathrm{e}^{2t}\right) + 2\left(c_1\mathrm{e}^t + c_2\mathrm{e}^{2t}\right) \\
&= 0
\end{aligned}$$

Implicit Solutions

A relation $G(x, y) = 0$, is an **implicit** solution if it determines one or more explicit solutions. For example, verify that $x^2 - \sin(x + y) = 1$ is an implicit solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\sec(x + y) - 1$. Differentiating $x^2 - \sin(x + y) = 1$ implicitly, we get:

$$\begin{aligned}
2x - \cos(x + y)\frac{\mathrm{d}}{\mathrm{d}x}(x + y) &= 0 \\
2x - \cos(x + y)(1 + \frac{\mathrm{d}y}{\mathrm{d}x}) &= 0 \\
2x - \cos(x + y) - \cos(x + y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
2x - \cos(x + y) &= \cos(x + y)\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x}{\cos(x + y)} - 1 \\
&= 2x\sec(x + y) - 1
\end{aligned}$$

Example

Show $x + y + e^{xy} = 0$ is an implicit solution to $(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0$.

$$\begin{aligned}
x + y + e^{xy} &= 0 \\
1 + \frac{dy}{dx} + e^{xy}\left[\frac{d}{dx}(xy)\right] &= 0 \\
1 + \frac{dy}{dx} + e^{xy}\left[y + x\frac{dy}{dx}\right] &= 0 \\
1 + \frac{dy}{dx} + ye^{xy} + xe^{xy}\frac{dy}{dx} &= 0 \\
1 + \left[1 + xe^{xy}\right]\frac{dy}{dx} + ye^{xy} &= 0 \\
(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} &= 0
\end{aligned}$$

Example

Show that $x^2 + y^2 - 25 = 0$ is an implicit solution to $\frac{dy}{dx} = f(x, y) = \frac{-x}{y}$ on $(-5, 5)$.

$$\begin{aligned}
x^2 + y^2 - 25 &= 0 \\
2x + 2y\frac{dy}{dx} &= 0 \\
y\frac{dy}{dx} &= -x \\
\frac{dy}{dx} &= \frac{-x}{y}
\end{aligned}$$

Alternatively, we can try doing this explicitly by solving for y:

$$\begin{aligned}
y^2 &= 25 - x^2 \\
y &= \pm\sqrt{25 - x^2} \\
\text{Choose } y &= \sqrt{25 - x^2} \\
&= (25 - x^2)^{\frac{1}{2}} \\
y' &= \frac{1}{2}\left(25 - x^2\right)^{-\frac{1}{2}}(-2x) \\
&= \frac{-x}{\sqrt{25 - x^2}} \\
&= \frac{-x}{y}
\end{aligned}$$

We want to find where $25 - x^2 > 0$, which tells us that $-5 < x < 5$.

Families of Solutions

Let's try to solve the following:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-y}{x} = f(x, y) \\
&= (-y)\frac{1}{x} \\
\frac{\mathrm{d}y}{y} &= -\frac{\mathrm{d}x}{x} \\
\int \frac{\mathrm{d}y}{y} &= -\int \frac{\mathrm{d}x}{x} \\
\ln|y| &= -\ln|x| + c
\end{aligned}$$

This is an **implicit** solution to our differential equation. We can also try to solve it explicitly by exponentiating both sides.

$$\begin{aligned}
\ln|y| &= -\ln|x| + c \\
\mathrm{e}^{\ln|y|} &= \mathrm{e}^{(-\ln|x| + c)} \\
|y| &= \mathrm{e}^{-\ln|x|}\mathrm{e}^{c} \\
|y| &= \mathrm{e}^{\ln|x^{-1}|}\mathrm{e}^{c} \\
&= \left|\frac{1}{x}\right|\mathrm{e}^{c}
\end{aligned}$$

If we let $k = \pm e^c$, then:

$$
\begin{aligned}
y &= \pm \frac{1}{x} e^c \\
&= \frac{k}{x}
\end{aligned}
$$

This is a **family** of one parameter k solutions. It is a general solution that contains an arbitrary constant k. Suppose we are given an initial value $y(1) = 5$, which lies on one of the curves.

$$
y = \frac{k}{x} \Longrightarrow 5 = \frac{k}{1} \Longrightarrow k = 5
$$

$y = \frac{5}{x}$ is a **particular** solution since there are no arbitrary constants.

$\mathbf{n^{th}}$ Order Initial Value Problems

Given $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ and n initial conditions:

$$\begin{aligned} y(x_\circ) &= y_0 \\ y'(x_\circ) &= y_1 \\ y''(x_\circ) &= y_2 \\ &\vdots \\ y^{(n-1)}(x_\circ) &= y_{n-1} \end{aligned}$$

We have an initial value problem. The solution over some interval I is called a particular solution.

Example

Verify that $y = c_1\cos(x)+c_2\sin(x)-\sin(2x)$ is a general solution to $y''+y = 3\sin(2x)$.

$$\begin{aligned} y &= c_1\cos(x) + c_2\sin(x) - \sin(2x) \\ y' &= -c_1\sin(x) + c_2\cos(x) - 2\cos(2x) \\ y'' &= -c_1\cos(x) - c_2\sin(x) + 4\sin(2x) \\ y'' + y &= 3\sin(2x) \end{aligned}$$

$$\begin{aligned} -c_1\cos(x) - c_2\sin(x) + 4\sin(2x) + c_1\cos(x) + c_2\sin(x) - \sin(2x) &= 3\sin(2x) \\ 3\sin(2x) &= 3\sin(2x) \end{aligned}$$

Find constants c_1 and c_2 so that the initial conditions $y(0) = 1$ and $y'(0) = 0$ are satisfied.

$$\begin{aligned} y &= c_1\cos(x) + c_2\sin(x) - \sin(2x) \\ y(0) &= 1 = c_1\cos(0) \\ c_1 &= 1 \\ y' &= -c_1\sin(x) + c_2\cos(x) - 2\cos(2x) \\ y'(0) &= 0 = 0 + c_2 - 2 \\ c_2 &= 2 \\ y &= \cos(x) + 2\sin(x) - \sin(2x) \end{aligned}$$

Example

Determine whether $y = c_1 e^{2t} + c_2 t e^{2t}$ is a general solution to $y'' - 4y' + 4y = 0$.

$$\begin{aligned}
y &= c_1 e^{2t} + c_2 t e^{2t} \\
y' &= 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} \\
y'' &= 4c_1 e^{2t} + 2c_2 e^{2t} + 2c_2 e^{2t} + 4c_2 t e^{2t} \\
y'' + 4y' + 2y &= 4c_1 e^{2t} + 2c_2 e^{2t} + 2c_2 e^{2t} + 4c_2 t e^{2t} - \\
&\quad 4\left(2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}\right) + \\
&\quad 4\left(c_1 e^{2t} + c_2 t e^{2t}\right) \\
&= 0
\end{aligned}$$

Find constants c_1 and c_2 such that the initial conditions $y(0) = 1$ and $y'(0) = 0$ are satisfied.

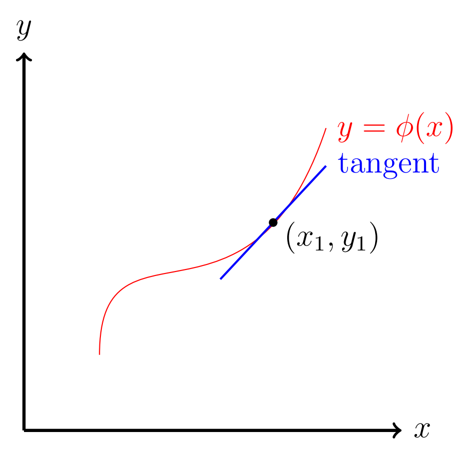
$$\begin{aligned}
y(0) &= 1 = c_1 \\
y'(0) &= 1 = 2 + c_2 \\
c_2 &= -1
\end{aligned}$$

Direction Fields of 1st Order Equations

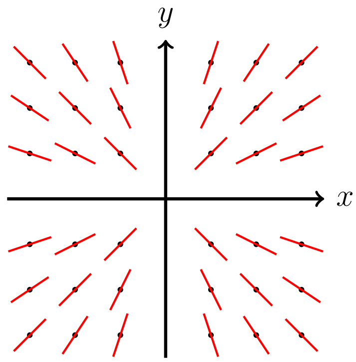
Consider the following:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

x is the independent variable and y is the dependent variable.



The idea is that the actual solution curve is $y = \phi(x)$. If we evaluate $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x_1, y_1)$, it gives the slope of the tangent at (x_1, y_1). We can construct a direction field with this. For example, if we have the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y}{x}$.



x	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
1	1	-1
1	2	-2
1	3	-3
1	-1	1
-1	1	1

Autonomous First-Order Equations

Autonomous first-order equations are equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$$

where the independent variable x does not appear in the equation. Consider the following:

$$f(y) = \frac{\mathrm{d}y}{\mathrm{d}x} = y - y^2$$

This is an autonomous equation. In contrast:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy = f(x, y)$$

This is not an autonomous equation. Equilibrium solutions are found by solving $f(y) = 0 = \frac{\mathrm{d}y}{\mathrm{d}x}$.

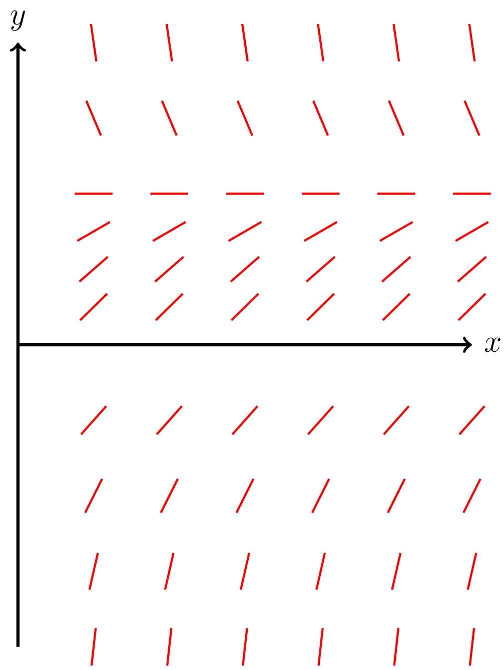
- If y is bounded by a critical point, then the graph of y approaches $y(t) = c$ as $t \to +\infty$.
- $\frac{\mathrm{d}y}{\mathrm{d}t}$ is either positive or negative except where $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$.
- y is monotonic, either increasing or decreasing, there is no oscillation.
- The graph of non-constant soultions never cross an equilibrium solution.

Example

A model for the velocity v at time t of an object falling under the influence of gravity in a viscous medium is given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 1 - \frac{v^3}{8}$$

Construct a direction field and determine the "terminal" velocity. Find values where $1 - \frac{v^3}{8} = 0$.



$$1 - \frac{v^3}{8} = 0 \Rightarrow v = 2$$

From this diagram, we can see all the families of solutions for this differential equation, but they all converge to $v = 2$ as $t \to \infty$.

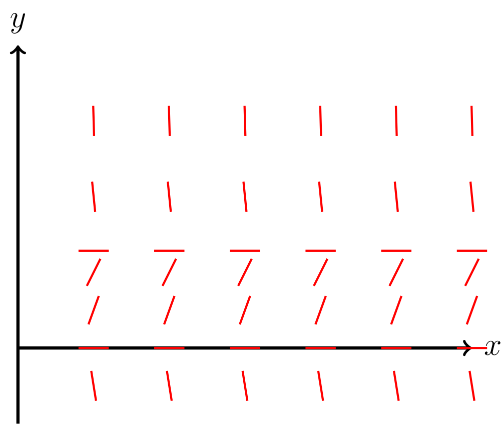
Example

Consider the following equation for a population (in thousands) of a certain species (Logistic Equation).

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 9p - 7p^2 = f(p)$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 9p - 7p^2 = 0 \Rightarrow p(9 - 7p) = 0$$

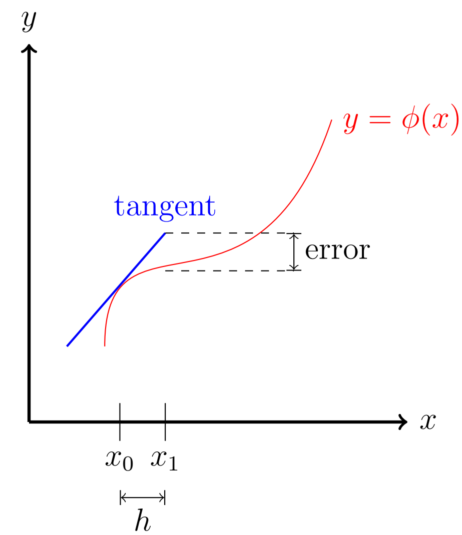
Our solutions are $p = 0$ or $p = \frac{9}{7}$.



$$\lim_{t \to \infty} = \frac{9}{7}$$

Euler's Method

Euler's method is also known as the tangent line method for finding numerical approximations to first-order initial value problems.



With this method, we also begin with some solution curve $y = \phi(x)$ and some given (x_0, y_0) that lies on the solution curve.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$
$$y(x_0) = y_0$$

$f(x_0, y_0)$ is the slope of the tangent to the curve at (x_0, y_0). Given (x_0, y_0), a single step h, and the slope $f(x_0, y_0)$:

$$y - y_0 = h\big[f(x_0, y_0)\big]$$
$$y_1 = y_0 + h\big[f(x_0, y_0)\big]$$

This is an iterative process defined by:

$$y_{n+1} = y_n + h\big[f(x_n, y_n)\big]$$

We determine the step size and iterate:

x_0	y_0
$x_1 = x_0 + h$	y_1
$x_2 = x_1 + h$	y_2
$\vdots$	$\vdots$
$x_{n+1} = x_n + h$	y_{n+1}

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2 = f(x, y)$$

If we take $y(1) = 1$ with $h = 0.1$ as our step size:

$x_0 = 1$	$y_0 = 1$
$x_1 = 1.1$	$y_1 = 1 + 0.1(2) = 1.2$
$x_2 = 1.2$	$y_2 = 1.2 + 0.1[(1.1)^2 + (1.2)^2] = 1.465$
$\vdots$	$\vdots$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech