Multivariable and Vector Calculus

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Review 2

Arc Length:

$$\int_C f\ \mathrm{d}S = \int_a^b f|r'(t)|\ \mathrm{d}t$$

$$\int_{C_{AB}} \vec{F} \cdot\ \mathrm{d}r = \int_C \vec{F} \cdot \vec{T}\ \mathrm{d}S = \int_a^b \vec{F} \cdot \overrightarrow{r'(t)}\ \mathrm{d}t$$

For $r(u, v) \quad (u, v) \in D$:

$$\iint_S f\ \mathrm{d}S = \iint_D f \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|\ \mathrm{d}u\ \mathrm{d}v$$

$$\iint_S \vec{F} \cdot\ \mathrm{d}S = \iint_S (\vec{F} \cdot \vec{r})\ \mathrm{d}S = \iint_D \vec{F} \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right)\ \mathrm{d}u\ \mathrm{d}v$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot\ \mathrm{d}S = \iint_S \mathrm{curl}\ F\ \mathrm{d}S = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right)\ \mathrm{d}u\ \mathrm{d}v$$

Greene's Theorem:

$$\iint_D Q_x - P_y\ \mathrm{d}A$$

Example

Compute the line integral:

$$\int_C x\ \mathrm{d}S$$

C is the curve $y = x^2$ from (0,0) to (1,1).

$$\begin{aligned} \overrightarrow{r(t)} &= \langle t, t^2 \rangle \\ \overrightarrow{r'(t)} &= \langle 1, 2t \rangle \\ \int_C x \ \mathrm{d}S &= \int_a^b f|r'(t)| \ \mathrm{d}t \\ &= \int_0^1 t\sqrt{1+4t^2} \ \mathrm{d}t \end{aligned}$$

Example

Compute the line integral:

$$\int_C \langle y, x+y^2 \rangle \cdot \ \mathrm{d}r$$

C is the curve $4x^2 + 9y^2 = 36$ in the counterclockwise orientation.

$$\overrightarrow{r(t)} = \left\langle \frac{6\cos(t)}{2}, \frac{6\sin(t)}{3} \right\rangle$$

$$\begin{aligned} \int_C \langle y, x+y^2 \rangle \cdot \ \mathrm{d}r &= \int_0^{2\pi} \langle 2\sin(t), 3\cos(t) + 4\sin^2(t) \rangle \cdot \langle -3\sin(t), 2\cos(t) \rangle \ \mathrm{d}t \\ &= \int_0^{2\pi} -6\sin^2(t) + 6\cos^2(t) + 8\sin^2(t)\cos(t) \ \mathrm{d}t \end{aligned}$$

Using Greene's Theorem:

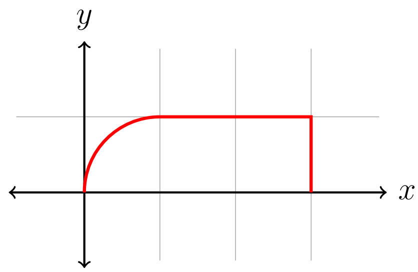
$$\begin{aligned} \int_C \langle y, x+y^2 \rangle \cdot \ \mathrm{d}r &= \iint_D Q_x - P_y \ \mathrm{d}A \\ &= \iint_D 1 - 1 \ \mathrm{d}A \\ &= 0 \end{aligned}$$

Example

Find the work done along the curve C of the function:

$$\int_C \langle 2xy, x^2 \rangle \cdot \ \mathrm{d}r$$

given that the curve C is:



Example

Evaluate:

$$\int_C \langle \mathrm{e}^y, x\mathrm{e}^y + \mathrm{e}^z, y\mathrm{e}^z \rangle \cdot \, \mathrm{d}r$$

Given C is the line from (0,2,0) to (4,0,3).

$$\begin{aligned} &f_x = \mathrm{e}^y \quad f = x\mathrm{e}^y + C_1(y,z) \\ &f_y = x\mathrm{e}^y + \mathrm{e}^z \quad f = x\mathrm{e}^y + y\mathrm{e}^z + C_2(x,z) \\ &f_z = y\mathrm{e}^z \quad f = y\mathrm{e}^z + C_3(x,y) \\ &f = x\mathrm{e}^y + y\mathrm{e}^z \end{aligned}$$

$$\int_C \langle \mathrm{e}^y, x\mathrm{e}^y + \mathrm{e}^z, y\mathrm{e}^z \rangle \cdot \, \mathrm{d}r = \int_{(0,2,0)}^{(4,0,3)} x\mathrm{e}^y + y\mathrm{e}^z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

Example

Give the parameterization of the region bounded by:

$$x^2 + y^2 + z^2 = 10 \quad z = 1 \quad z = 3$$

$$\overrightarrow{r(R,\theta)} = \langle R\cos\theta, R\sin\theta, \sqrt{10 - R^2} \rangle \quad 1 \le R \le 3 \quad 0 \le \theta \le 2\pi$$

Example

Find the surface area of the region bounded by:

$$z = \sqrt{x^2 + y^2} \quad y = x \quad y = x^2$$

$$\overrightarrow{r(x,y)} = \langle x, y, \sqrt{x^2+y^2}\rangle$$

$$\iint_S 1 \ \mathrm{d}S = \int_0^1 \int_{x^2}^x \left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}x$$

$$= \int_0^1 \int_{x^2}^x \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{2x}{2\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{2y}{2\sqrt{x^2+y^2}} \end{vmatrix} \ \mathrm{d}y \ \mathrm{d}x$$

$$= \int_0^1 \int_{x^2}^x \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} \ \mathrm{d}y \ \mathrm{d}x$$

$$= \int_0^1 \int_{x^2}^x \sqrt{2} \ \mathrm{d}y \ \mathrm{d}x$$

Example

Evaluate:

$$\iint_S \langle x^2, xy, z\rangle \cdot \ \mathrm{d}S$$

where S is bounded by $z = x^2 + y^2$ below $z = 1$ with upwards orientation.

$$\overrightarrow{r(R,\theta)} = \langle R\cos\theta, R\sin\theta, R^2\rangle$$

$$\left|\frac{\partial r}{\partial R} \times \frac{\partial r}{\partial \theta}\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2R \\ -R\sin\theta & R\cos\theta & 0 \end{vmatrix}$$

$$= \langle -2R^2\cos\theta - 2R^2\sin\theta, R\rangle$$

$$\iint_S \langle x^2, xy, z\rangle \cdot \ \mathrm{d}S = \int_0^{2\pi} \int_0^1 \langle R^2\cos^2\theta, R^2\sin\theta\cos\theta, R^2\rangle \cdot \left|\frac{\partial r}{\partial R} \times \frac{\partial r}{\partial \theta}\right| \ \mathrm{d}R \ \mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^1 -2R^4\cos^3\theta - 4R^4\sin^2\theta\cos\theta + R^3 \ \mathrm{d}R \ \mathrm{d}\theta$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech