Multivariable and Vector Calculus

Alvin Lin

August 2017 - December 2017

Review 1

$$\vec{a} \cdot \vec{b} = a_1b_2 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}|\cos\theta$$

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$\vec{a} \cdot \vec{b} = 0 \equiv \vec{a} \perp \vec{b}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\vec{a} \times \vec{b} = \vec{0} \equiv \vec{a} \parallel \vec{b}$$

$$\text{Volume of a parellelepiped} = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\text{Line} = \begin{cases} x &= x_\circ + ta_1 \\ y &= y_\circ + ta_2 \\ z &= z_\circ + ta_3 \end{cases}$$

$$\text{Plane} = n_1(x - x_\circ) + n_2(y - y_\circ) + n_3(z - z_\circ) = 0$$

Practice Problem

Find the angle between the diagonal of a cube and the edge of the base starting at (0,0,0).

$$\cos\alpha = \frac{\langle 1,1,1\rangle \cdot \langle 0,1,0\rangle}{|\langle 1,1,1\rangle||\langle 0,1,0\rangle|} = \frac{1}{\sqrt{3}}$$

Practice Problem

Find x such that $\langle 1,x,2\rangle \perp \langle x,3,4\rangle$.

$$\begin{aligned} \langle 1,x,2\rangle \cdot \langle x,3,4\rangle &= 0 \\ x + 3x + 8 &= 0 \\ x &= -2 \end{aligned}$$

Is there such an x that the two vectors are parallel?

$$\begin{aligned} c\langle 1,x,2\rangle &= \langle x,3,r\rangle \\ c &= x \\ cx &= 3 \\ 2c &= 4 \end{aligned}$$

This system of linear equations is inconsistent, therefore the two vectors are not parallel.

Practice Problem

Given the planes $\Pi_1 : x - 2y + z = 4$ and $\Pi_2 : 2x - 2x - z = 6$. Find the line of intersection between the planes $l = \Pi_1 \cap \Pi_2$.

If we let $x = 0$:

$$\begin{cases} -2y + z &= 4 \\ -2y - z &= 6 \end{cases}$$

$$y = -\frac{5}{2} \quad z = -1 \quad P_1 = (0, -\frac{5}{2}, 1)$$

If we let $y = 0$:

$$\begin{cases} x + z &= 4 \\ 2x - z &= 6 \end{cases}$$

$$x = \frac{10}{3} \quad z = \frac{2}{3} \quad P_2 = (\frac{10}{3}, 0, \frac{2}{3})$$

$$\overrightarrow{P_1P_2} = \langle \frac{10}{3}, \frac{5}{2}, -\frac{1}{3} \rangle$$

$$l = \begin{cases} x &= \frac{10}{3} + t\frac{10}{3} \\ x &= t\frac{5}{2} \\ z &= \frac{2}{3} - t\frac{1}{3} \end{cases}$$

Find the angle between Π_1, Π_2.

$$\cos\alpha = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{\langle 1, -2, 1\rangle \cdot \langle 2, -2, -1\rangle}{\sqrt{6}\sqrt{9}} = \frac{5}{3\sqrt{6}}$$

Practice Problem

Given $\Pi_1 : x - 2y + z = 4$ and $l : x - 2 = y + 5 = z - 4 = t$, find $P = l \cap \Pi$.

$$\begin{aligned} x - 2y + z &= 4 \\ (t + 2) - 2(t - 5) + (t + 4) &= 4 \\ 16 &\neq 4 \end{aligned}$$

Since there is no solution, the line l does not intersect the plane.
Find the distance between l and Π. Pick any point P_0 on the plane and any point P_1 on the line.

$$\begin{aligned} P_0 &= (4, 0, 0) \\ P_1 &= (2, -5, 4) \\ dist(l, \Pi) &= comp_{\vec{n}}\overrightarrow{P_0P_1} \\ &= \frac{\overrightarrow{P_0P_1} \cdot \vec{n}}{|\vec{n}|} \\ &= \frac{\langle -2, -5, 4\rangle \cdot \langle 1, -2, 1\rangle}{|\langle 1, -2, 1\rangle|} \\ &= \frac{14}{\sqrt{6}} \end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech