

# Multivariable and Vector Calculus: Homework 12

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## Section 16.9

### Exercise 7

Use the Divergence Theorem to calculate the surface integral  $\iint_S F \cdot dS$ ; that is, calculate the flux of  $F$  across  $S$ .

$$F(x, y, z) = 3xy^2\hat{i} + xe^z\hat{j} + z^3\hat{k}$$

$S$  is the surface of the solid bounded by the cylinder  $y^2 + z^2 = 1$  and the planes  $x = -1$  and  $x = 2$ .

$$\begin{aligned}\iint_S F \cdot dS &= \iiint_E \operatorname{div} F \, dV \\ \operatorname{div} F &= 3y^2 + 0 + 3z^2 \\ 3 \iiint_E y^2 + z^2 \, dV &= 3 \int_{-1}^2 \int_0^{2\pi} \int_0^1 (r \cos \theta)^2 + (r \sin \theta)^2 r \, dr \, d\theta \, dz \\ &= 3 \int_{-1}^2 \int_0^{2\pi} \int_0^1 r^3 (\cos^2 \theta + \sin^2 \theta) \, dr \, d\theta \, dz \\ &= 3 \int_{-1}^2 \int_0^{2\pi} \left[ \frac{r^4}{4} \right]_0^1 \, d\theta \, dz \\ &= \frac{3}{4} \int_{-1}^2 \int_0^{2\pi} d\theta \, dz \\ &= \frac{3}{4} \int_{-1}^2 \left[ \theta \right]_0^{2\pi} \, dz \\ &= \frac{6\pi}{4} \int_{-1}^2 dz \\ &= \frac{3\pi}{2} \left[ z \right]_{-1}^2 \\ &= \frac{9\pi}{2}\end{aligned}$$

### Exercise 11

Use the Divergence Theorem to calculate the surface integral  $\iint_S F \cdot dS$ ; that is, calculate the flux of  $F$  across  $S$ .

$$F(x, y, z) = (2x^3 + y^3)\hat{i} + (y^3 + z^3)\hat{j} + 3y^2z\hat{k}$$

$S$  is the surface of the solid bounded by the paraboloid  $z = 1 - x^2 - y^2$  and the  $xy$ -plane.

$$\begin{aligned}
 \iint_S F \, dS &= \iiint_E \operatorname{div} F \, dV \\
 \operatorname{div} F &= 6x^2 + 3y^2 + 3y^2 \\
 &= 6(x^2 + y^2) \\
 \iiint_E 6(x^2 + y^2) \, dV &= 6 \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^2 r \, dz \, dr \, d\theta \\
 &= 6 \int_0^{2\pi} \int_0^1 r^3 \left[ z \right]_0^{1-r^2} \, dr \, d\theta \\
 &= 6 \int_0^{2\pi} \int_0^1 r^3(1-r^2) \, dr \, d\theta \\
 &= 6 \int_0^{2\pi} \left[ \frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 \, d\theta \\
 &= 6 \int_0^{2\pi} \frac{1}{4} - \frac{1}{6} \, d\theta \\
 &= \frac{6}{12} \int_0^{2\pi} \, d\theta \\
 &= \frac{2\pi}{2} = \pi
 \end{aligned}$$

### Exercise 17

Use the Divergence Theorem to evaluate  $\iint_S F \cdot dS$ , where  $F(x, y, z) = z^2 x \hat{i} + (\frac{1}{3}y^3 + \tan(z)) \hat{j} + (x^2 z + y^2) \hat{k}$  and  $S$  is the top half of the sphere  $x^2 + y^2 + z^2 = 1$ . Note that  $S$  is not a closed surface. First compute integrals over  $S_1$  and  $S_2$ , where  $S_1$  is the disk  $x^2 + y^2 \leq 1$ , oriented downward, and  $S_2 = S \cup S_1$ .

$$\begin{aligned}
 \iint_S F \, dS &= \iiint_E \operatorname{div} F \, dV \\
 \operatorname{div} F &= z^2 + y^2 + x^2 \\
 \iint_{S_1} F \, dS &= \iint_D F \cdot (-\hat{k}) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA \\
 &= - \iint_D (x^2 z + y^2) \sqrt{1} \, dA \\
 &= - \int_0^{2\pi} \int_0^1 (r \cos \theta)^2 (0) + (r \sin \theta)^2 r \, dr \, d\theta \\
 &= - \int_0^{2\pi} \sin^2 \theta \int_0^1 r^3 \, dr \, d\theta \\
 &= - \int_0^{2\pi} \sin^2 \theta \left[ \frac{r^4}{4} \right]_0^1 \, d\theta
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4} \int_0^{2\pi} \frac{1 - \cos(2\theta)}{2} d\theta \\
&= -\frac{1}{8} \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} \\
&= -\frac{2\pi}{8} = -\frac{\pi}{4}
\end{aligned}$$

### Exercise 25

Prove each identity, assuming that  $S$  and  $E$  satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

$$\iint_S \vec{a} \cdot \vec{n} \, dS = 0$$

where  $\vec{a}$  is a constant vector.

$$\begin{aligned}
&\operatorname{div} \vec{a} = 0 \\
&\iint_S \vec{a} \cdot \vec{n} \, dS = \iiint_E 0 \, dV = 0
\end{aligned}$$

### Exercise 27

Prove each identity, assuming that  $S$  and  $E$  satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

$$\iint_S \operatorname{curl} F \cdot d\vec{S} = 0$$

$$\begin{aligned}
\iint_S \operatorname{curl} F \cdot d\vec{S} &= \iiint_E \operatorname{div}(\operatorname{curl} F) \, dV \\
&= \iiint_E \nabla \cdot (\nabla \times F) \, dV \\
&= \iiint_E 0 \, dV \\
&= 0
\end{aligned}$$

## Section 16.8

### Exercise 5

Use Stokes' Theorem to evaluate  $\iint_S \operatorname{curl} F \cdot d\vec{S}$ .

$$F(x, y, z) = xyz\hat{i} + xy\hat{j} + x^2yz\hat{k}$$

$S$  consists of the top and the four sides (but not the bottom) of the cube with vertices  $(\pm 1, \pm 1, \pm 1)$ , oriented outward.

$$\begin{aligned}
 \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \iint_H \text{curl } \mathbf{F} \cdot \hat{\mathbf{k}} \, dS \\
 &= \iint_H \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dS \\
 &= \iint_H y - zx \, dS \\
 &= \iint_S y + x \, dS \\
 &= \int_{-1}^1 \int_{-1}^1 y + x \, dx \, dy \\
 &= \int_{-1}^1 \left[ yx + \frac{x^2}{2} \right]_{-1}^1 \, dy \\
 &= \int_{-1}^1 2y \, dy \\
 &= 0
 \end{aligned}$$

### Exercise 9

Use Stokes' Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ . In each case  $C$  is oriented counterclockwise as viewed from above.

$$\mathbf{F}(x, y, z) = xy\hat{i} + yz\hat{j} + zx\hat{k}$$

$C$  is the boundary of the part of the paraboloid  $1 - x^2 - y^2$  in the first octant.

$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \\
 &= \iint_D -P\left(\frac{\partial g}{\partial x}\right) - Q\left(\frac{\partial g}{\partial y}\right) + R \, dA \\
 &= \iint_D y(-2x) - (-z)(-2y) - x \, dA \\
 &= \iint_D -2xy - 2yz - x \, dA \\
 &= \iint_D -2xy - 2y(1 - x^2 - y^2) - x \, dA \\
 &= \iint_D -2xy - 2y + 2yx^2 + 2y^3 - x \, dA \\
 &= \iint_D -2y(x + 1 - x^2 - y^2) - x \, dA
 \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \int_0^1 (-2r \cos \theta (r \sin \theta + 1 - r^2) - r \cos \theta) r \, dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 -2r^3 \cos \theta \sin \theta - 2r \cos \theta + 2r^4 \cos \theta - r^2 \cos \theta \, dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 -r^3 \sin 2\theta - 2r \cos \theta + 2r^4 \cos \theta - r^2 \cos \theta \, dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[ \frac{-r^4 \sin 2\theta}{4} - \frac{2r^3 \sin \theta}{3} + \frac{2r^5 \sin \theta}{5} - \frac{r^3 \cos \theta}{3} \right]_0^1 d\theta \\
&= \int_0^{\frac{\pi}{2}} -\frac{\sin 2\theta}{4} - \frac{2 \sin \theta}{3} + \frac{2 \sin \theta}{5} - \frac{\cos \theta}{3} d\theta \\
&= \left[ \frac{1}{2} \frac{\cos 2\theta}{4} + \frac{2 \cos \theta}{3} - \frac{2 \cos \theta}{5} - \frac{\sin \theta}{3} \right]_0^{\frac{\pi}{2}} \\
&= \left( -\frac{1}{8} + 0 - 0 - \frac{1}{3} \right) - \left( \frac{1}{8} + \frac{2}{3} - \frac{2}{5} - 0 \right) \\
&= -\frac{17}{20}
\end{aligned}$$

### Exercise 15

Verify that Stokes' Theorem is true for the given vector field  $F$  and surface  $S$ .

$$F(x, y, z) = y\hat{i} + z\hat{j} + x\hat{k}$$

$S$  is the hemisphere  $x^2 + y^2 + z^2 = 1$ ,  $y \geq 0$ , oriented in the direction of the positive  $y$ -axis.

?

### Exercise 17

A particle moves along line segments from the origin to the points  $(1,0,0)$ ,  $(1,2,1)$ ,  $(0,2,1)$ , and back to the origin under the influence of the force field

$$F(x, y, z) = z^2\hat{i} + 2xy\hat{j} + 4y^2\hat{k}$$

Find the work done.

$$\begin{aligned}\int_C F \cdot dr &= \iint_S \text{curl } F \, dS \\ &= \iint_D -P\left(\frac{\partial g}{\partial x}\right) - Q\left(\frac{\partial g}{\partial y}\right) + R \, dA \\ &= \iint_D -8y\left(\frac{\partial g}{\partial x}\right) - 2z\left(\frac{\partial g}{\partial y}\right) + 2y \, dA \\ &= \iint_D 0 - z + 2y \, dA \\ &= \iint_D 0 - \frac{y}{2} + 2y \, dA \\ &= \iint_D \frac{3y}{2} \, dA \\ &= \frac{3}{2} \int_0^1 \int_0^2 y \, dy \, dx \\ &= \frac{3}{2} \int_0^1 2 \, dx \\ &= 3\end{aligned}$$

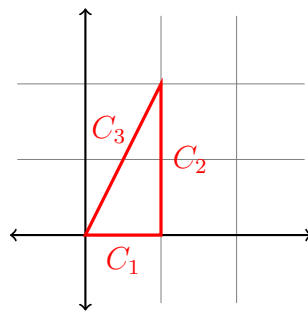
## Section 16.4

### Exercise 3

Evaluate the line integral directly and using Green's Theorem.

$$\oint_C xy \, dx + x^2 y^3 \, dy$$

$C$  is the triangle with vertices  $(0,0)$ ,  $(1,0)$ , and  $(1,2)$



$$\begin{aligned}
\oint_C xy \, dx + x^2 y^3 \, dy &= \int_{C_1} xy \, dx + x^2 y^3 \, dy + \int_{C_2} xy \, dx + x^2 y^3 \, dy + \int_{C_3} xy \, dx + x^2 y^3 \, dy \\
&= \int_{C_1} 0 \, dx + 0 + \int_{C_2} 0 + y^3 \, dy + \int_{C_3} \frac{y}{2} y \frac{dy}{2} + \left(\frac{y}{2}\right)^2 y^3 \, dy \\
&= 0 + \int_0^2 y^3 \, dy + \int_2^0 \frac{y^2}{4} \, dy + \frac{y^5}{4} \, dy \\
&= 0 + \left[\frac{y^4}{4}\right]_0^2 + \left[\frac{y^3}{12} + \frac{y^6}{24}\right]_2^0 \\
&= 4 - \frac{2}{3} - \frac{8}{3} \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\oint_C xy \, dx + x^2 y^3 \, dy &= \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \\
&= \int_0^1 \int_0^{2x} -x + 2xy^3 \, dy \, dx \\
&= \int_0^1 \left[ -xy + \frac{xy^4}{2} \right]_0^{2x} \, dx \\
&= \int_0^1 -2x^2 + 8x^5 \, dx \\
&= \left[ -\frac{2x^3}{3} + \frac{8x^6}{6} \right]_0^1 \\
&= -\frac{2}{3} + \frac{8}{6} \\
&= \frac{2}{3}
\end{aligned}$$

### Exercise 7

Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C (y + e^{\sqrt{x}}) \, dx + (2x + \cos(y^2)) \, dy$$

$C$  is the boundary of the region enclosed by the parabolas  $y = x^2$  and  $x = y^2$ .

$$\begin{aligned}
\oint_C P \, dx + Q \, dy &= \iint_D \left( \frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) \, dA \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} (2 - 1) \, dy \, dx \\
&= \int_0^1 \left[ y \right]_{x^2}^{\sqrt{x}} \, dx \\
&= \int_0^1 \sqrt{x} - x^2 \, dx \\
&= \left[ \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right]_0^1 \\
&= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}
\end{aligned}$$

### Exercise 13

Use Green's Theorem to evaluate  $\int_C F \cdot dr$ .

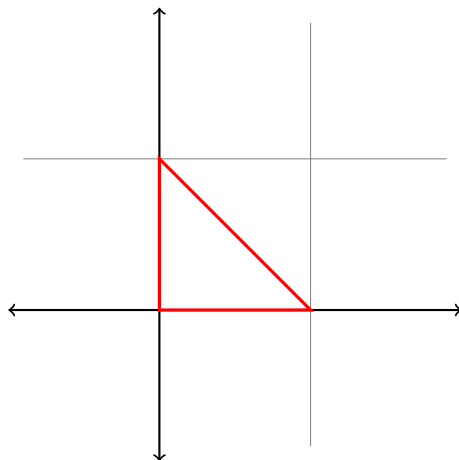
$$F(x, y) = \langle y - \cos(y), x \sin(y) \rangle$$

$C$  is the circle  $(x - 3)^2 + (y + 4)^2 = 4$  oriented clockwise.

$$\begin{aligned} \oint_C P \, dx + Q \, dy &= \iint_D \left( \frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) \, dA \\ &= \iint_D \sin(y) + 1 - \sin(y) \, dx \, dy \\ &= \int_0^2 \int_0^{2\pi} r \, d\theta \, dr \\ &= \int_0^2 \left[ r\theta \right]_0^{2\pi} \, dr \\ &= \int_0^2 2\pi r \, dr \\ &= \left[ \pi r^2 \right]_0^2 \\ &= 4\pi \end{aligned}$$

### Exercise 17

Use Green's Theorem to find the work done by the force  $F(x, y) = x(x + y)\hat{i} + xy^2\hat{j}$  in moving a particle from the origin along the x-axis to  $(1,0)$ , then along the line segment to  $(0,1)$ , and then back to the origin along the y-axis.





$$\begin{aligned}
\oint_C P \, dx + Q \, dy &= \iint_D \left( \frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) \, dA \\
&= \iint_D y^2 - x \, dA \\
&= \int_0^1 \int_0^{1-x} y^2 - x \, dy \, dx \\
&= \int_0^1 \left[ \frac{y^3}{3} - xy \right]_0^{1-x} \, dx \\
&= \int_0^1 \frac{(1-x)^3}{3} - x(1-x) \, dx \\
&= \int_0^1 \frac{1 - 3x + 3x^2 - x^3}{3} - x + x^2 \, dx \\
&= \int_0^1 \frac{1}{3} - x + x^2 - \frac{x^3}{3} - x + x^2 \, dx \\
&= \int_0^1 -\frac{x^3}{3} + 2x^2 - 2x + \frac{1}{3} \, dx \\
&= \left[ -\frac{x^4}{12} + \frac{2x^3}{3} - x^2 + \frac{x}{3} \right]_0^1 \\
&= -\frac{1}{12} + \frac{2}{3} - 1 + \frac{1}{3} \\
&= -\frac{1}{12}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)