Multivariable and Vector Calculus: Homework 11

Alvin Lin

August 2016 - December 2016

Section 16.3

Exercise 3

Determine whether or not F is a conservative vector field. If it is, find a function f such that $F = \nabla f$.

$$F(x,y) = (xy + y^2)\hat{i} + (x^2 + 2y)\hat{j}$$

$$\begin{aligned}
F(x,y) &= P\hat{i} + Q\hat{j} \\
\frac{\partial P}{\partial y} &= x + 2y \\
\frac{\partial Q}{\partial x} &= 2x \\
\frac{\partial P}{\partial y} &\neq \frac{\partial Q}{\partial x}
\end{aligned}$$

Exercise 5

Determine whether or not F is a conservative vector field. If it is, find a function f such that $F = \nabla f$.

$$F(x,y) = y^2 \mathrm{e}^{xy}\hat{i} + (1 + xy)\mathrm{e}^{xy}\hat{j}$$

$$\begin{aligned}
F(x,y) &= P\hat{i} + Q\hat{j} \\
\frac{\partial P}{\partial y} &= 2y\mathrm{e}^{xy} + xy^2\mathrm{e}^{xy} \\
\frac{\partial Q}{\partial x} &= y\mathrm{e}^{xy} + (1+xy)y\mathrm{e}^{xy} \\
&= 2y\mathrm{e}^{xy} + xy^2\mathrm{e}^{xy} \\
\int y^2\mathrm{e}^{xy} &= y\mathrm{e}^{xy} + h(y) \\
(1+xy)\mathrm{e}^{xy} + h'(y) &= (1+xy)\mathrm{e}^{xy} \\
h'(y) &= 0 \\
h(y) &= c \\
f(x,y) &= y\mathrm{e}^{xy} + c
\end{aligned}$$

Exercise 13

Find a function f such that $F = \nabla f$ and use it to evaluate $\int_C F \cdot \, \mathrm{d}r$ along the given curve C.

$$F(x,y) = x^2y^3\hat{\imath} + x^3y^2\hat{\jmath} \quad C : \overrightarrow{r(t)} = \langle t^3 - 2t, t^3 + 2t \rangle \quad 0 \le t \le 1$$

$$\begin{aligned}
f(x,y) &= \frac{1}{3}x^3y^3 + c \\
\overrightarrow{r(0)} &= \langle 0, 0 \rangle \\
\overrightarrow{r(1)} &= \langle -1, 3 \rangle \\
\int_C F \, \mathrm{d}r &= f(-1,3) - f(0,0) \\
&= \frac{1}{3}(-1)^3(3)^3 - 0 \\
&= -9
\end{aligned}$$

Exercise 17

Find a function f such that $F = \nabla f$ and use it to evaluate $\int_C F \cdot \, \mathrm{d}r$ along the given curve C.

$$F(x,y,z) = yz\mathrm{e}^{xz}\hat{\imath} + \mathrm{e}^{xz}\hat{\jmath} + xy\mathrm{e}^{xz}\hat{k}$$

$$C : r(\vec{t}) = (t^2+1)\hat{\imath} + (t^2-1)\hat{\jmath} + (t^2-2t)\hat{k} \quad 0 \le t \le 2$$

$$\begin{aligned}
f_x &= yz\mathrm{e}^{xz} \\
f_y &= \mathrm{e}^{xz} \\
f_z &= xy\mathrm{e}^{xz} \\
f &= \int f_x \, \mathrm{d}x \\
&= \int yz\mathrm{e}^{xz} \, \mathrm{d}x \\
&= yz\frac{\mathrm{e}^{xz}}{z} + h(y,z) \\
f_y &= yz\mathrm{e}^{xz} = yz\mathrm{e}^{xz} + h_y(y,z) \\
h_y(y,z) &= 0 \\
h(y,z) &= g(z) \\
f_z &= xy\mathrm{e}^{xz} + g'(z) = xy\mathrm{e}^{xz} \\
g'(z) &= 0 \\
g(z) &= h(y,z) = c \\
f(x,y) &= y\mathrm{e}^{xz} + c \\
\overrightarrow{r(0)} &= \langle 1, -1, 0 \rangle \\
\overrightarrow{r(2)} &= \langle 5, 3, 0 \rangle \\
\int_C F \cdot \, \mathrm{d}r &= f(5,3,0) - f(1,-1,0) \\
&= 3\mathrm{e}^{(5)(0)} - (-1)\mathrm{e}^{(1)(0)} \\
&= 3 + 1 \\
&= 4
\end{aligned}$$

Exercise 21

Suppose you're asked to determine the curve that requires the least work for a force field F to move a particle from one point to another point. You decide to check first whether F is conservative, and indeed turns out that it is. How would you reply to the request?

Because F is conservative, the path taken to F does not matter since $\int_C F \cdot \, \mathrm{d}r$ is $f(\overrightarrow{r(b)}) - f(\overrightarrow{r(a)})$ and only the endpoints a and b of the curve matter.

Exercise 23

Find the work done by the force field F in moving an object from P to Q.

$$F(x, y) = x^3\hat{\imath} + y^3\hat{\jmath} \quad P(1, 0), Q(2, 2)$$

$$\begin{aligned}
f_x &= x^3 \\
f_y &= y^3 \\
f(x, y) &= \int f_x \, \mathrm{d}x \\
&= \int x^3 \, \mathrm{d}x \\
&= \frac{1}{4}x^4 + h(y) \\
f_y(x, y) &= h'(y) = y^3 \\
h(y) &= \int y^3 \, \mathrm{d}y \\
&= \frac{1}{4}y^4 + c \\
f(x, y) &= \frac{1}{4}x^4 + (\frac{1}{4}y^4 + c) \\
W &= \int f \cdot \, \mathrm{d}r \\
&= f(2, 2) - f(1, 1) \\
&= \frac{1}{4}(32) - \frac{1}{4}(1) \\
&= \frac{31}{4}
\end{aligned}$$

Exercise 31

Determine whether or not the given set is open, connected, and/or simply connected.

$$\{(x, y) \mid 0 < y < 3\}$$

This defines all points within the square from (0,0) to (3,3) but excluding the boundary of the square. This region is open since it does not include those boundary points. The set is connected since all points in the set can be joined by a path entirely composed of points into the set.

Exercise 33

Determine whether or not the given set is open, connected, and/or simply connected.

$$\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4 \quad y \geq 0\}$$

This defines the area between the semicircles above the x-axis with radius 1 and 2. This set is not an open set because it contains its boundary points. It is a connected set since all points can be joined with a path inside the set.

Section 16.6

Exercise 2

Determine whether the points P and Q lie on the given surface.

$$\overrightarrow{r(u,v)} = \langle 1+u-v, u+v^2, u^2-v^2 \rangle \quad P(1,2,1), Q(2,3,3)$$

$$\begin{aligned} 1+u-v &= 1 \\ u+v^2 &= 2 \\ u^2-v^2 &= 1 \\ 1+u-v-(u+v^2) &= 1-2 \\ v^2+v-2 &= 0 \\ (v+2)(v-1) &= 0 \\ v &= -2 \quad v=1 \\ u &= -2 \quad u=1 \\ u^2-v^2 &\neq 1 \end{aligned}$$

The point P does not lie on the surface.

$$\begin{aligned} 1+u-v &= 2 \\ u+v^2 &= 3 \\ u^2-v^2 &= 3 \\ 1+u-v-(u+v^2) &= 2-3 \\ v^2+v-2 &= 0 \\ v &= -2 \quad v=1 \\ u &= -1 \quad u=2 \\ u^2-v^2 &= 1 \end{aligned}$$

The point Q lies on the surface.

Exercise 19

Find a parametric representation for the plane through the origin that contains the vectors $\hat{\imath}-\hat{\jmath}$ and $\hat{\jmath}-\hat{k}$.

$$\begin{aligned} \overrightarrow{r(u,v)} &= \vec{p} + (\hat{\imath}-\hat{\jmath})u + (\hat{\jmath}-\hat{k})v \\ &= u\hat{\imath} - u\hat{\jmath} + v\hat{\jmath} - v\hat{k} \\ &= u\hat{\imath} + (v-u)\hat{\jmath} - v\hat{k} \\ x = u \quad y &= v-u \quad z = -v \end{aligned}$$

Exercise 33

Find an equation of the tangent plane to the given parametric surface at the specified point.

$$x = u + v \quad y = 3u^2 \quad z = u - v \quad (2, 3, 0)$$

$$\begin{aligned}
u + v &= 2 \\
3u^2 &= 3 \\
u - v &= 0 \\
u &= \pm 1 \\
v &= 1 \\
u &= v = 1 \\
\overrightarrow{r(u,v)} &= (u+v)\hat{\imath} + 3u^2\hat{\jmath} + (u-v)\hat{k} \\
\vec{r_u} &= \hat{\imath} + 6u\hat{\jmath} + \hat{k} \\
\vec{r_v} &= \hat{\imath} - \hat{k} \\
\vec{n} &= \vec{r_u} \times \vec{r_v} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 6u & 1 \\ 1 & 0 & -1 \end{vmatrix} \\
&= -6u\hat{\imath} + 2\hat{\jmath} + -6u\hat{k} \\
0 &= (\langle x, y, z\rangle - P) \cdot \vec{n} \\
&= (x-2)(-6(1)) + (y-3)(2) + (z-0)(-6(1)) \\
&= -6x + 12 + 2y - 6 - 6z \\
6x - 2y + 6z &= 6 \\
3x - y + 3z &= 3
\end{aligned}$$

Exercise 39

Find the area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

$$\begin{aligned}
z &= 6 - 3x - 2y \\
\frac{\partial z}{\partial x} &= -3 \\
\frac{\partial z}{\partial y} &= -2
\end{aligned}$$

$$
\begin{aligned}
A(S) &= \iint_D \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\ \mathrm{d}A \\
&= \iint_D \sqrt{1+9+4}\ \mathrm{d}A \\
&= \sqrt{14}\iint_D\ \mathrm{d}A \\
&= \sqrt{14}\int_0^2\int_0^{3-\frac{3x}{2}}\ \mathrm{d}y\ \mathrm{d}x \\
&= \sqrt{14}\int_0^2 3-\frac{3x}{2}\ \mathrm{d}x \\
&= \sqrt{14}\left[3x-\frac{3x^2}{4}\right]_0^2 \\
&= 3\sqrt{14}
\end{aligned}
$$

Exercise 45

Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

$$
\begin{aligned}
z &= xy \\
\frac{\partial z}{\partial x} &= y \\
\frac{\partial z}{\partial y} &= x \\
A(S) &= \iint_D \sqrt{1+y^2+x^2}\ \mathrm{d}A \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1+y^2+x^2}\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}r\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_0^{2\pi}\left[\frac{1}{3}(1+r^2)^{\frac{3}{2}}\right]_0^1\ \mathrm{d}\theta \\
&= \frac{1}{3}\int_0^{2\pi} 2\sqrt{2}-1\ \mathrm{d}\theta \\
&= \frac{2\sqrt{2}-1}{3}\Big[\theta\Big]_0^{2\pi} \\
&= \frac{2\pi(2\sqrt{2}-1)}{3}
\end{aligned}
$$

Section 16.7

Exercise 5

Evaluate the surface integral.

$$\iint_S (x+y+z)\ \mathrm{d}S$$

S is the parallelogram with parametric equations $x = u+v, y = u-v, z = 1+2u+v, 0 \leq u \leq 2, 0 \leq v \leq 1$.

$$\begin{aligned}
\overrightarrow{r(u,v)} &= (u+v)\hat{\imath} + (u-v)\hat{\jmath} + (1+2u+v)\hat{k} \\
\vec{r_u} &= \hat{\imath} + \hat{\jmath} + 2\hat{k} \\
\vec{r_v} &= \hat{\imath} - \hat{\jmath} + \hat{k} \\
\iint_S (x+y+z)\ \mathrm{d}S &= \iint_D f(\overrightarrow{r(u,v)})|\vec{r_u} \times \vec{r_v}|\ \mathrm{d}A \\
&= \int_0^2 \int_0^1 (4u+1+v)|\vec{r_u} \times \vec{r_v}|\ \mathrm{d}v\ \mathrm{d}u \\
&= \sqrt{14} \int_0^2 \left[4uv + v + \frac{v^2}{2}\right]_0^1 \mathrm{d}u \\
&= \sqrt{14} \int_0^2 4u + 1 + \frac{1}{2}\ \mathrm{d}u \\
&= \sqrt{14} \left[\frac{4u^2}{2} + u + \frac{u}{2}\right]_0^1 \\
&= \sqrt{14}(8 + 3 - (0+0)) \\
&= 11\sqrt{14}
\end{aligned}$$

Exercise 11

Evaluate the surface integral.

$$\iint_S x\ \mathrm{d}S$$

S is the triangular region with vertices $(1,0,0), (0,-2,0), (0,0,4)$.

$$\begin{aligned}
\frac{x}{1} + \frac{y}{-2} + \frac{z}{4} &= 1 \\
4x - 2y + z &= 4 \\
z &= 4 - 4x + 2y \\
\frac{\partial z}{\partial x} &= -4 \\
\frac{\partial z}{\partial y} &= 2
\end{aligned}$$

$$\begin{aligned}
\iint_S x \text{ d}S &= \iint_D f(x,y,z)\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \text{ d}A \\
&= \iint_D \sqrt{1+16+4}x \text{ d}A \\
&= \sqrt{21}\int_0^1 \int_{2x-2}^0 x \text{ d}y \text{ d}x \\
&= \sqrt{21}\int_0^1 \Big[xy\Big]_{2x-2}^0 \text{ d}x \\
&= \sqrt{21}\int_0^1 -2x^2 + 2x \text{ d}x \\
&= \sqrt{21}\left[\frac{2x^2}{2} - \frac{2x^3}{3}\right]_0^1 \\
&= \sqrt{21}\left(\frac{2}{2} - \frac{2}{3}\right) \\
&= \frac{\sqrt{21}}{3}
\end{aligned}$$

Exercise 15

Evaluate the surface integral.

$$\iint_S x \text{ d}S$$

S is the surface $y = x^2 + 4z, 0 \le x \le 1, 0 \le z \le 1$.

$$\begin{aligned}
\frac{\partial y}{\partial x} &= 2x \\
\frac{\partial y}{\partial z} &= 4 \\
\iint_S x \text{ d}S &= \iint_D x\sqrt{1+16+4x^2} \text{ d}x \text{ d}z \\
&= \int_0^1 \int_0^1 x\sqrt{17+4x^2} \text{ d}x \text{ d}z \\
&= \int_0^1 \left[\frac{1}{12}(4x^2+17)^{\frac{3}{2}}\right]_0^1 \text{ d}z \\
&= \int_0^1 \frac{1}{12}(21^{\frac{3}{2}} - 17^{\frac{3}{2}}) \text{ d}z \\
&= \frac{21^{\frac{3}{2}} - 17^{\frac{3}{2}}}{12}
\end{aligned}$$

Exercise 17

Evaluate the surface integral.

$$\iint_S (x^2z + y^2z) \text{ d}S$$

S is the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$.

$$
\begin{aligned}
x^2 + y^2 + z^2 = 4 \equiv \rho = 2 \\
x &= 2 \sin\phi \cos\theta \\
y &= 2 \sin\phi \sin\theta \\
z &= 2 \cos\phi \\
\vec{r_\phi} &= 2\cos\phi\cos\theta\hat{i} + 2\cos\phi\cos\theta\hat{j} - 2\sin\phi\hat{k} \\
\vec{r_\phi} &= 2\sin\phi\sin\theta\hat{i} + 2\sin\phi\cos\theta\hat{j} + 0\hat{k} \\
|\vec{r_\phi} \times \vec{r_\theta}| &= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix} \right\| \\
&= 4\sin\phi \\
\iint_S (x^2 x + y^2 z)\ \mathrm{d}S &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (8\sin^2\phi\cos\phi\cos^2\theta + 8\sin^2\phi\cos\phi\sin^2\theta)(4\sin\phi)\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 32\sin^3\phi\cos\theta\ \mathrm{d}\phi\ \mathrm{d}\theta \\
&= 32 \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \sin^3\theta\phi\cos\theta\ \mathrm{d}\theta \\
&= 32(2\pi)(\frac{1}{4}) \\
&= 16\pi
\end{aligned}
$$

Exercise 19

Evaluate the surface integral.

$$\iint_S xz\ \mathrm{d}S$$

S is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$.

$$
\begin{aligned}
S_1 : y^2 + z^2 &= 9 \\
\overrightarrow{r(x,\theta)} &= x\hat{i} + 3\cos\theta\hat{j} + 3\sin\theta\hat{k} \\
\vec{r_x} &= \hat{i} \\
\vec{r_\theta} &= -3\sin\theta\hat{j} + 3\cos\theta\hat{k} \\
|\vec{r_x} \times \vec{r_\theta}| &= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & -3\sin\theta & 3\cos\theta \end{vmatrix} \right\| \\
&= \sqrt{(-3\cos\theta)^2 + (-3\sin\theta)^2} \\
&= 3 \\
\iint_{S_1} xz\ \mathrm{d}S &= \int_0^{2\pi} \int_0^{5-3\cos\theta} x(3\sin\theta)(3)\ \mathrm{d}x\ \mathrm{d}\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2} 9x^2 \sin\theta \right]_0^{5-3\cos\theta} \mathrm{d}\theta \\
&= 0
\end{aligned}
$$

$$S_2 : x = 0$$

$$\iint_{S_2} xz \; \mathrm{d}S = \iint_{S_2} 0 \; \mathrm{d}S$$
$$= 0$$

$$S_3 : x + y = 5$$
$$z = 0$$
$$\iint_{D} xz \; \mathrm{d}S = 0$$

$$\iint_{S} xz \; \mathrm{d}S = 0 + 0 + 0 = 0$$

Exercise 23

Evaluate the surface integral $\iint_S F \cdot \; \mathrm{d}S$ for the given vector field F and the oriented surface S. In other words, find the flux of F across S. For closed surfaces, use the positive (outward) orientation.

$$F(x, y, z) = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$$

S is the part of paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1, 0 \le y \le 1$ and has upward orientation.

$$F = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$$
$$= xy\hat{\imath} + yz\hat{\jmath} + xz\hat{k}$$
$$z = g(x, y) = 4 - x^2 - y^2$$
$$\frac{\partial g}{\partial x} = -2x$$
$$\frac{\partial g}{\partial y} = -2y$$

$$
\begin{aligned}
\iint_S F \cdot \, \mathrm{d}S &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) \, \mathrm{d}A \\
&= \iint_D -(xy)(-2x) - (yz)(-2y) + xz \, \mathrm{d}A \\
&= \int_0^1 \int_0^1 2x^2y + 2y^2z + xz \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \int_0^1 2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2) \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \int_0^1 2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \left[\frac{2x^2y}{2} + \frac{8y^3}{3} - \frac{2x^2y^3}{3} - \frac{2y^5}{5} + 4xy - x^3y - \frac{xy^3}{3}\right]_0^1 \mathrm{d}x \\
&= \int_0^1 x^2 + \frac{8}{3} - \frac{2x^2}{3} - \frac{2}{5} + 4x - x^3 - \frac{x}{3} \, \mathrm{d}x \\
&= \int_0^1 -x^3 + \frac{x^2}{3} + \frac{11x}{3} + \frac{34}{15} \, \mathrm{d}x \\
&= \left[-\frac{x^4}{4} + \frac{x^3}{9} + \frac{11x^2}{6} + \frac{34x}{15}\right]_0^1 \\
&= -\frac{1}{4} + \frac{1}{9} + \frac{11}{6} + \frac{34}{15} \\
&\approx 3.96111
\end{aligned}
$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech