Multivariable and Vector Calculus: Homework 9

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Section 15.7

Exercise 7

Identify the surface whose equation is given.

$$r^2 + z^2 = 4$$

$$x^2 + y^2 + z^2 = 4$$

A sphere centered at the origin with radius 2.

Exercise 8

Identify the surface whose equation is given.

$$\begin{aligned} r &= 2\sin\theta \\ r^2 &= 2r\sin\theta \\ x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y &= 0 \\ x^2 + y^2 - 2y + 1 &= 1 \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$

A cylinder with radius 1 centered at (0,1).

Exercise 17

Use cylindrical coordinates to evaluate $\iiint_E \sqrt{x^2+y^2}\ \mathrm{d}V$ where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

$$\begin{aligned}
\iiint_E \sqrt{x^2+y^2}\ \mathrm{d}V &= \int_{-5}^{4}\int_{0}^{4}\int_{0}^{2\pi} \sqrt{r^2}r\ \mathrm{d}z\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_{-5}^{4}\int_{0}^{4} \Big[zr^2\Big]_0^{2\pi}\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_{-5}^{4}\int_{0}^{4} 2\pi r^2\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_{-5}^{4} \left[\frac{2\pi r^3}{3}\right]_0^4\ \mathrm{d}\theta \\
&= \frac{2\pi}{3}\int_{-5}^{4} 64\ \mathrm{d}\theta \\
&= \frac{128\pi}{3}\Big[\theta\Big]_{-5}^{4} \\
&= \frac{128\pi(9)}{3} \\
&= 384\pi
\end{aligned}$$

Exercise 21

Use cylindrical coordinates to evaluate $\iiint_E x^2\ \mathrm{d}V$ where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$ and below the cone $z^2=4x^2+4y^2$.

$$\begin{aligned}
\iiint_E x^2\ \mathrm{d}V &= \int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{\sqrt{4r^2}} r^2\sin^2(\theta) r\ \mathrm{d}z\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_{0}^{2\pi}\int_{0}^{1} \Big[zr^3\cos^2\theta\Big]_0^{2r}\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_{0}^{2\pi}\int_{0}^{1} 2r^4\cos^2\theta\ \mathrm{d}r\ \mathrm{d}\theta \\
&= 2\int_{0}^{2\pi} \cos^2\theta\left[\frac{r^5}{5}\right]_0^1\ \mathrm{d}\theta \\
&= \frac{2}{5}\int_{0}^{2\pi} \frac{1+\cos(2\theta)}{2}\ \mathrm{d}\theta \\
&= \frac{1}{5}\left[\theta+\frac{\sin(2\theta)}{2}\right]_0^{2\pi} \\
&= \frac{1}{2}\left(2\pi+\frac{\sin(4\pi)}{2}-0-\frac{\sin(0)}{2}\right) \\
&= \frac{2\pi}{5}
\end{aligned}$$

Exercise 29

Evaluate the integral by changing to cylindrical coordinates.

$$\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz \text{ d}z \text{ d}x \text{ d}y &= \int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r\cos\theta zr \text{ d}z \text{ d}r \text{ d}\theta \\
&= \int_{0}^{2\pi}\int_{0}^{2} r^2\cos\theta\Big[z\Big]_{r}^{2} \text{ d}r \text{ d}\theta \\
&= \int_{0}^{2\pi}\cos\theta\int_{0}^{2} r^2(2-r) \text{ d}r \text{ d}\theta \\
&= \int_{0}^{2\pi}\cos\theta\int_{0}^{2} 2r^2-r^3 \text{ d}r \text{ d}\theta \\
&= \int_{0}^{2\pi}\cos\theta\left[\frac{2r^3}{3}-\frac{r^4}{4}\right]_{0}^{2} \text{ d}\theta \\
&= \int_{0}^{2\pi}\cos\theta\left(\frac{16}{3}-4\right) \text{ d}\theta \\
&= \frac{4}{3}\Big[-\sin\theta\Big]_{0}^{2\pi} \\
&= \frac{4}{3}\Big(-\sin(2\pi)+\sin(0)\Big) \\
&= 0
\end{aligned}$$

Section 15.8

Exercise 7

Identify the surface whose equation is given.

$$\rho\cos\phi = 1$$

$$z = 1$$

The plane $z = 1$.

Exercise 13

Sketch the solid described by the given inequalities.

$$2 \le \rho \le 4, 0 \le \phi \le \frac{\pi}{3}, 0 \le \theta \le \pi$$

The region between half-spheres of radius of 2 and 4 contained between 0 and $\frac{\pi}{3}$.

Exercise 17

Sketch the solid whose volume is given by the integral and evaluate the integral. The slice of a quarter sphere of radius 3 contained between 0 and $\frac{\pi}{6}$.

$$\begin{aligned}
\int_0^{\frac{\pi}{6}}\int_0^{\frac{\pi}{2}}\int_0^{3}\rho^2\sin\phi\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi &= \int_0^{\frac{\pi}{6}}\sin\phi\int_0^{\frac{\pi}{2}}\left[\frac{\rho^3}{3}\right]_0^3\mathrm{d}\theta\ \mathrm{d}\phi\\
&= 9\int_0^{\frac{\pi}{6}}\sin\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\ \mathrm{d}\phi\\
&= 9\int_0^{\frac{\pi}{6}}\sin\phi\left[\theta\right]_0^{\frac{\pi}{2}}\mathrm{d}\phi\\
&= \frac{9\pi}{2}\int_0^{\frac{\pi}{6}}\sin\phi\ \mathrm{d}\phi\\
&= \frac{9\pi}{2}\left[-\cos\phi\right]_0^{\frac{\pi}{6}}\\
&= \frac{9\pi}{2}\Big(-\cos(\frac{\pi}{6})+\cos(0)\Big)\\
&= \frac{9\pi}{2}\left(-\frac{\sqrt{3}}{2}+1\right)\\
&= \frac{18\pi-9\pi\sqrt{3}}{4}
\end{aligned}$$

Exercise 21

Use spherical coordinates to evaluate $\iiint_B(x^2+y^2+z^2)^2\ \mathrm{d}V$ where B is the ball with center the origin and radius 5.

$$\begin{aligned}
\iiint_B(x^2+y^2+z^2)^2\ \mathrm{d}V &= \int_0^{\pi}\int_0^{2\pi}\int_0^{5}(\rho^2)^2\rho^2\sin\phi\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi\\
&= \int_0^{\pi}\sin\phi\int_0^{2\pi}\left[\frac{\rho^7}{7}\right]_0^5\mathrm{d}\theta\ \mathrm{d}\phi\\
&= \frac{78125}{7}\int_0^{\pi}\sin\phi\left[\theta\right]_0^{2\pi}\mathrm{d}\phi\\
&= \frac{78125}{7}\int_0^{\pi}2\pi\sin\phi\ \mathrm{d}\phi\\
&= \frac{2\pi(78125)}{7}\left[-\cos\phi\right]_0^{\pi}\\
&= \frac{2\pi(78125)}{7}\Big(-\cos\pi+\cos(0)\Big)\\
&= \frac{312500\pi}{7}
\end{aligned}$$

Exercise 23

Use spherical coordinates to evaluate $\iiint_E (x^2+y^2)\ \mathrm{d}V$, where E lies between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

$$\begin{aligned}
\iiint_E (x^2+y^2)\ \mathrm{d}V &= \int_0^{\pi}\int_0^{2\pi}\int_2^3 (\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2\rho^2\sin\phi\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \int_0^{\pi}\int_0^{2\pi}\int_2^3 \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)\rho^2\sin\phi\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \int_0^{\pi}\sin^3\phi\int_0^{2\pi}\left[\frac{\rho^5}{5}\right]_2^3\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \frac{211}{5}\int_0^{\pi}\sin^3\phi\int_0^{2\pi}\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \frac{422\pi}{5}\int_0^{\pi}\sin^3\phi\ \mathrm{d}\phi \\
&= \frac{422\pi}{5}\int_0^{\pi}\sin\phi(1-\cos^2\phi)\ \mathrm{d}\phi \\
&= \frac{422\pi}{5}\int_0^{\pi}\sin\phi-\sin\phi\cos^2\phi\ \mathrm{d}\phi \\
&= \frac{422\pi}{5}\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi} \\
&= \frac{422\pi}{5}\left(-\cos\pi+\frac{1}{3}\cos^3\pi+\cos(0)-\frac{1}{3}\cos^3(0)\right) \\
&= \frac{422\pi}{5}\left(1-\frac{1}{3}+1-\frac{1}{3}\right) \\
&= \frac{422\pi}{5}\frac{4}{3} \\
&= \frac{1688\pi}{15}
\end{aligned}$$

Exercise 25

Use spherical coordinates to evaluate $\iiint_E x\mathrm{e}^{x^2+y^2+z^2}\ \mathrm{d}V$, where E is the portion of the unit ball $x^2+y^2+z^2 \leq 1$ that lies in the first octant.

$$\begin{aligned}
\iiint_E x\mathrm{e}^{x^2+y^2+z^2}\ \mathrm{d}V &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho\sin\phi\cos\theta\mathrm{e}^{\rho^2}\rho^2\sin\phi\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \int_0^{\frac{\pi}{2}}\sin^2\phi\int_0^{\frac{\pi}{2}}\cos\theta\int_0^1\rho^3\mathrm{e}^{\rho^2}\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2\phi\int_0^{\frac{\pi}{2}}\cos\theta\Big[\mathrm{e}^{x^2}x^2-\mathrm{e}^{x^2}\Big]_0^1\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2\phi\int_0^{\frac{\pi}{2}}\cos\theta(\mathrm{e}-\mathrm{e}+1)\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2\phi\Big[\sin\theta\Big]_0^{\frac{\pi}{2}}\ \mathrm{d}\phi \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2\phi\ \mathrm{d}\phi \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}}\phi\ \mathrm{d}\phi
\end{aligned}$$

Exercise 27

Use spherical coordinates to find the volume of the part of the ball $\rho \leq a$ that lies between the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.

$$\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^a \rho^2\sin\phi\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\phi\int_0^{2\pi}\int_0^a\rho^2\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\phi\int_0^{2\pi}\left[\frac{\rho^3}{3}\right]_0^a\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \frac{a^3}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\phi\int_0^{2\pi}\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \frac{2\pi a^3}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\phi\ \mathrm{d}\phi \\
&= \frac{2\pi a^3}{3}\Big[-\cos\phi\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= \frac{\pi a^3\sqrt{3}-\pi a^3}{3}
\end{aligned}$$

Exercise 29a

Use spherical coordinates to find the volume of the solid that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 4\cos\phi$.

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_0^{4\cos\phi} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi &= \int_0^{\frac{\pi}{3}} \sin\phi \int_0^{2\pi} \left[\frac{\rho^3}{3}\right]_0^{4\cos\phi} \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \frac{64}{3} \int_0^{\frac{\pi}{3}} \sin\phi \cos^3\phi \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \frac{128\pi}{3} \int_0^{\frac{\pi}{3}} \sin\phi \cos^3\phi \, \mathrm{d}\phi \\
&= \frac{128\pi}{3} \left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{3}} \\
&= \frac{32\pi}{3} \left(-\cos^4\frac{\pi}{3} + \cos^4(0)\right) \\
&= \frac{32\pi}{3} \left(\frac{15}{16}\right) \\
&= 10\pi
\end{aligned}$$

Exercise 35

Use cylindrical or spherical coordinates, whichever seems more appropriate, to find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

$$\begin{aligned}
V &= \iiint_E \mathrm{d}V \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{1} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \int_0^{\frac{\pi}{4}} \sin\phi \int_0^{2\pi} \left[\frac{\rho^3}{3}\right]_0^{1} \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \frac{1}{3} \int_0^{\frac{\pi}{4}} \sin\phi \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \sin\phi \, \mathrm{d}\phi \\
&= \frac{2\pi}{3} \left[-\cos\phi\right]_0^{\frac{\pi}{4}} \\
&= \frac{2\pi - \sqrt{2}\pi}{3}
\end{aligned}$$

This solid is symmetrical about x and y.

$$\begin{aligned}
centroid_z &= \frac{1}{V}\iiint_E z \ \mathrm{d}V \\
&= \frac{3}{2\pi - \sqrt{2}\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{1} \rho\cos\phi\rho^2\sin\phi \ \mathrm{d}\rho \ \mathrm{d}\theta \ \mathrm{d}\phi \\
&= \frac{3}{2\pi - \sqrt{2}\pi}\int_0^{\frac{\pi}{4}} \cos\phi\sin\phi \int_0^{2\pi} \left[\frac{\rho^4}{4}\right]_0^1 \mathrm{d}\theta \ \mathrm{d}\phi \\
&= \frac{3}{8(2\pi - \sqrt{2}\pi)}\int_0^{\frac{\pi}{4}} \cos\phi\sin\phi \int_0^{2\pi} \ \mathrm{d}\theta \ \mathrm{d}\phi \\
&= \frac{6}{8(2 - \sqrt{2})}\int_0^{\frac{\pi}{4}} \cos\phi\sin\phi \ \mathrm{d}\phi \\
&= \frac{6}{8(2 - \sqrt{2})}\int_0^{\frac{\pi}{4}} \frac{\sin 2\phi}{2} \ \mathrm{d}\phi \\
&= \frac{3}{16(2 - \sqrt{2})}\Big[- \cos 2\phi\Big]_0^{\frac{\pi}{4}} \\
&= \frac{3}{16(2 - \sqrt{2})}\left(- \cos\frac{\pi}{2} + \cos(0)\right) \\
&= \frac{3}{16(2 - \sqrt{2})}(2) \\
&= \frac{3}{8(2 - \sqrt{2})}
\end{aligned}$$

Centroid: $(0, 0, \frac{3}{16-8\sqrt{2}})$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech