

Multivariable and Vector Calculus: Homework 9

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Section 15.7

Exercise 7

Identify the surface whose equation is given.

$$r^2 + z^2 = 4$$

$$x^2 + y^2 + z^2 = 4$$

A sphere centered at the origin with radius 2.

Exercise 8

Identify the surface whose equation is given.

$$r = 2 \sin \theta$$

$$r^2 = 2r \sin \theta$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y - 1)^2 = 1$$

A cylinder with radius 1 centered at (0,1).

Exercise 17

Use cylindrical coordinates to evaluate $\iiint_E \sqrt{x^2 + y^2} \, dV$ where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

$$\begin{aligned}
 \iiint_E \sqrt{x^2 + y^2} \, dV &= \int_{-5}^4 \int_0^4 \int_0^{2\pi} \sqrt{r^2} r \, dz \, dr \, d\theta \\
 &= \int_{-5}^4 \int_0^4 \left[zr^2 \right]_0^{2\pi} \, dr \, d\theta \\
 &= \int_{-5}^4 \int_0^4 2\pi r^2 \, dr \, d\theta \\
 &= \int_{-5}^4 \left[\frac{2\pi r^3}{3} \right]_0^4 \, d\theta \\
 &= \frac{2\pi}{3} \int_{-5}^4 64 \, d\theta \\
 &= \frac{128\pi}{3} \left[\theta \right]_{-5}^4 \\
 &= \frac{128\pi(9)}{3} \\
 &= 384\pi
 \end{aligned}$$

Exercise 21

Use cylindrical coordinates to evaluate $\iiint_E x^2 \, dV$ where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and below the cone $z^2 = 4x^2 + 4y^2$.

$$\begin{aligned}
 \iiint_E x^2 \, dV &= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4r^2}} r^2 \sin^2(\theta) r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 \left[zr^3 \cos^2 \theta \right]_0^{2r} \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 2r^4 \cos^2 \theta \, dr \, d\theta \\
 &= 2 \int_0^{2\pi} \cos^2 \theta \left[\frac{r^5}{5} \right]_0^1 \, d\theta \\
 &= \frac{2}{5} \int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} \, d\theta \\
 &= \frac{1}{5} \left[\theta + \frac{\sin(2\theta)}{2} \right]_0^{2\pi} \\
 &= \frac{1}{5} \left(2\pi + \frac{\sin(4\pi)}{2} - 0 - \frac{\sin(0)}{2} \right) \\
 &= \frac{2\pi}{5}
 \end{aligned}$$

Exercise 29

Evaluate the integral by changing to cylindrical coordinates.

$$\begin{aligned}\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^2 xz \, dz \, dx \, dy &= \int_0^{2\pi} \int_0^2 \int_r^2 r \cos \theta z r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 r^2 \cos \theta \left[z \right]_r^2 \, dr \, d\theta \\ &= \int_0^{2\pi} \cos \theta \int_0^2 r^2(2-r) \, dr \, d\theta \\ &= \int_0^{2\pi} \cos \theta \int_0^2 2r^2 - r^3 \, dr \, d\theta \\ &= \int_0^{2\pi} \cos \theta \left[\frac{2r^3}{3} - \frac{r^4}{4} \right]_0^2 \, d\theta \\ &= \int_0^{2\pi} \cos \theta \left(\frac{16}{3} - 4 \right) \, d\theta \\ &= \frac{4}{3} \left[-\sin \theta \right]_0^{2\pi} \\ &= \frac{4}{3} \left(-\sin(2\pi) + \sin(0) \right) \\ &= 0\end{aligned}$$

Section 15.8

Exercise 7

Identify the surface whose equation is given.

$$\rho \cos \phi = 1$$

$$z = 1$$

The plane $z = 1$.

Exercise 13

Sketch the solid described by the given inequalities.

$$2 \leq \rho \leq 4, 0 \leq \phi \leq \frac{\pi}{3}, 0 \leq \theta \leq \pi$$

The region between half-spheres of radius of 2 and 4 contained between 0 and $\frac{\pi}{3}$.

Exercise 17

Sketch the solid whose volume is given by the integral and evaluate the integral. The slice of a quarter sphere of radius 3 contained between 0 and $\frac{\pi}{6}$.

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi &= \int_0^{\frac{\pi}{6}} \sin \phi \int_0^{\frac{\pi}{2}} \left[\frac{\rho^3}{3} \right]_0^3 d\theta \, d\phi \\ &= 9 \int_0^{\frac{\pi}{6}} \sin \phi \int_0^{\frac{\pi}{2}} d\theta \, d\phi \\ &= 9 \int_0^{\frac{\pi}{6}} \sin \phi \left[\theta \right]_0^{\frac{\pi}{2}} d\phi \\ &= \frac{9\pi}{2} \int_0^{\frac{\pi}{6}} \sin \phi \, d\phi \\ &= \frac{9\pi}{2} \left[-\cos \phi \right]_0^{\frac{\pi}{6}} \\ &= \frac{9\pi}{2} \left(-\cos\left(\frac{\pi}{6}\right) + \cos(0) \right) \\ &= \frac{9\pi}{2} \left(-\frac{\sqrt{3}}{2} + 1 \right) \\ &= \frac{18\pi - 9\pi\sqrt{3}}{4}\end{aligned}$$

Exercise 21

Use spherical coordinates to evaluate $\iiint_B (x^2 + y^2 + z^2)^2 \, dV$ where B is the ball with center the origin and radius 5.

$$\begin{aligned}\iiint_B (x^2 + y^2 + z^2)^2 \, dV &= \int_0^\pi \int_0^{2\pi} \int_0^5 (\rho^2)^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^\pi \sin \phi \int_0^{2\pi} \left[\frac{\rho^7}{7} \right]_0^5 d\theta \, d\phi \\ &= \frac{78125}{7} \int_0^\pi \sin \phi \left[\theta \right]_0^{2\pi} d\phi \\ &= \frac{78125}{7} \int_0^\pi 2\pi \sin \phi \, d\phi \\ &= \frac{2\pi(78125)}{7} \left[-\cos \phi \right]_0^\pi \\ &= \frac{2\pi(78125)}{7} \left(-\cos \pi + \cos(0) \right) \\ &= \frac{312500\pi}{7}\end{aligned}$$

Exercise 23

Use spherical coordinates to evaluate $\iiint_E (x^2 + y^2) \, dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

$$\begin{aligned}\iiint_E (x^2 + y^2) \, dV &= \int_0^\pi \int_0^{2\pi} \int_2^3 (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^\pi \int_0^{2\pi} \int_2^3 \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^\pi \sin^3 \phi \int_0^{2\pi} \left[\frac{\rho^5}{5} \right]_2^3 \, d\theta \, d\phi \\ &= \frac{211}{5} \int_0^\pi \sin^3 \phi \int_0^{2\pi} \, d\theta \, d\phi \\ &= \frac{422\pi}{5} \int_0^\pi \sin^3 \phi \, d\phi \\ &= \frac{422\pi}{5} \int_0^\pi \sin \phi (1 - \cos^2 \phi) \, d\phi \\ &= \frac{422\pi}{5} \int_0^\pi \sin \phi - \sin \phi \cos^2 \phi \, d\phi \\ &= \frac{422\pi}{5} \left[-\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^\pi \\ &= \frac{422\pi}{5} \left(-\cos \pi + \frac{1}{3} \cos^3 \pi + \cos(0) - \frac{1}{3} \cos^3(0) \right) \\ &= \frac{422\pi}{5} \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) \\ &= \frac{422\pi}{5} \frac{4}{3} \\ &= \frac{1688\pi}{15}\end{aligned}$$

Exercise 25

Use spherical coordinates to evaluate $\iiint_E x e^{x^2+y^2+z^2} dV$, where E is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

$$\begin{aligned}\iiint_E x e^{x^2+y^2+z^2} dV &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho \sin \phi \cos \theta e^{\rho^2} \rho^2 \sin \phi d\rho d\theta d\phi \\ &= \int_0^{\frac{\pi}{2}} \sin^2 \phi \int_0^{\frac{\pi}{2}} \cos \theta \int_0^1 \rho^3 e^{\rho^2} d\rho d\theta d\phi \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 \phi \int_0^{\frac{\pi}{2}} \cos \theta \left[e^{x^2} x^2 - e^{x^2} \right]_0^1 d\theta d\phi \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 \phi \int_0^{\frac{\pi}{2}} \cos \theta (e - e + 1) d\theta d\phi \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 \phi \left[\sin \theta \right]_0^{\frac{\pi}{2}} d\phi \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 \phi d\phi \\ &= \frac{1}{4} \int_0^{\frac{\pi}{2}} \phi d\phi\end{aligned}$$

Exercise 27

Use spherical coordinates to find the volume of the part of the ball $\rho \leq a$ that lies between the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^{2\pi} \int_0^a \rho^2 \sin \phi d\rho d\theta d\phi &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \phi \int_0^{2\pi} \int_0^a \rho^2 d\rho d\theta d\phi \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \phi \int_0^{2\pi} \left[\frac{\rho^3}{3} \right]_0^a d\theta d\phi \\ &= \frac{a^3}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \phi \int_0^{2\pi} d\theta d\phi \\ &= \frac{2\pi a^3}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \phi d\phi \\ &= \frac{2\pi a^3}{3} \left[-\cos \phi \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{\pi a^3 \sqrt{3} - \pi a^3}{3}\end{aligned}$$

Exercise 29a

Use spherical coordinates to find the volume of the solid that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 4 \cos \phi$.

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_0^{4 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi &= \int_0^{\frac{\pi}{3}} \sin \phi \int_0^{2\pi} \left[\frac{\rho^3}{3} \right]_0^{4 \cos \phi} \, d\theta \, d\phi \\ &= \frac{64}{3} \int_0^{\frac{\pi}{3}} \sin \phi \cos^3 \phi \int_0^{2\pi} \, d\theta \, d\phi \\ &= \frac{128\pi}{3} \int_0^{\frac{\pi}{3}} \sin \phi \cos^3 \phi \, d\phi \\ &= \frac{128\pi}{3} \left[-\frac{\cos^4 \phi}{4} \right]_0^{\frac{\pi}{3}} \\ &= \frac{32\pi}{3} \left(-\cos^4 \frac{\pi}{3} + \cos^4(0) \right) \\ &= \frac{32\pi}{3} \left(\frac{15}{16} \right) \\ &= 10\pi \end{aligned}$$

Exercise 35

Use cylindrical or spherical coordinates, whichever seems more appropriate, to find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

$$\begin{aligned} V &= \iiint_E dV \\ &= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^{\frac{\pi}{4}} \sin \phi \int_0^{2\pi} \left[\frac{\rho^3}{3} \right]_0^1 \, d\theta \, d\phi \\ &= \frac{1}{3} \int_0^{\frac{\pi}{4}} \sin \phi \int_0^{2\pi} \, d\theta \, d\phi \\ &= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \sin \phi \, d\phi \\ &= \frac{2\pi}{3} \left[-\cos \phi \right]_0^{\frac{\pi}{4}} \\ &= \frac{2\pi - \sqrt{2}\pi}{3} \end{aligned}$$

This solid is symmetrical about x and y.

$$\begin{aligned}
 centroid_z &= \frac{1}{V} \iiint_E z \, dV \\
 &= \frac{3}{2\pi - \sqrt{2}\pi} \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^1 \rho \cos \phi \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= \frac{3}{2\pi - \sqrt{2}\pi} \int_0^{\frac{\pi}{4}} \cos \phi \sin \phi \int_0^{2\pi} \left[\frac{\rho^4}{4} \right]_0^1 \, d\theta \, d\phi \\
 &= \frac{3}{8(2\pi - \sqrt{2}\pi)} \int_0^{\frac{\pi}{4}} \cos \phi \sin \phi \int_0^{2\pi} d\theta \, d\phi \\
 &= \frac{6}{8(2 - \sqrt{2})} \int_0^{\frac{\pi}{4}} \cos \phi \sin \phi \, d\phi \\
 &= \frac{6}{8(2 - \sqrt{2})} \int_0^{\frac{\pi}{4}} \frac{\sin 2\phi}{2} \, d\phi \\
 &= \frac{3}{16(2 - \sqrt{2})} \left[-\cos 2\phi \right]_0^{\frac{\pi}{4}} \\
 &= \frac{3}{16(2 - \sqrt{2})} \left(-\cos \frac{\pi}{2} + \cos(0) \right) \\
 &= \frac{3}{16(2 - \sqrt{2})} (2) \\
 &= \frac{3}{8(2 - \sqrt{2})}
 \end{aligned}$$

Centroid: $(0, 0, \frac{3}{16-8\sqrt{2}})$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech