

Multivariable and Vector Calculus: Homework 8

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Section 15.3

Exercise 9

Evaluate the given integral by changing to polar coordinates.

$$\iint_R \sin(x^2 + y^2) \, dA$$

where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3.

$$\begin{aligned} \iint_R \sin(x^2 + y^2) \, dA &= \int_0^{\frac{\pi}{2}} \int_1^3 \sin(r^2) r \, dr \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[-\frac{\cos(r^2)}{2} \right]_1^3 \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} (-\cos(9) + \cos(1)) \, d\theta \\ &= \frac{1}{2} \left[\cos(1)\theta - \cos(9)\theta \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi(\cos(1) - \cos(9))}{4} \\ &= \frac{\pi}{4}(\cos(1) - \cos(9)) \end{aligned}$$

Exercise 15

Use a double integral to find the area of one loop of the rose $r = \cos(3\theta)$.

$$\begin{aligned} \iint_D \, dA &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos(3\theta)} r \, dr \, d\theta \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2} \right]_0^{\cos(3\theta)} \, d\theta \\ &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(3\theta) \, d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1 + \cos(6\theta))}{2} d\theta \\
&= \frac{1}{4} \left[\theta - \frac{\sin(6\theta)}{6} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{1}{4} \left(\frac{\pi}{4} - \frac{1}{6} \sin\left(\frac{6\pi}{4}\right) + \frac{\pi}{4} + \frac{1}{6} \sin\left(-\frac{6\pi}{4}\right) \right) \\
&= \frac{\pi}{8}
\end{aligned}$$

Exercise 17

Use a double integral to find the area of the region inside the circle $(x - 1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$.

$$\begin{aligned}
(x - 1)^2 + y^2 &= 1 \\
x^2 - 2x + 1 + y^2 &= 1 \\
x^2 + y^2 - 2x &= 0 \\
r^2 - 2r \cos \theta &= 0 \\
r &= 2 \cos \theta \\
x^2 + y^2 &= 1 \\
r &= 1 \\
r = 1 &= 2 \cos \theta \\
\theta = -\frac{\pi}{3} \quad \theta &= \frac{\pi}{3} \\
\iint_D dA &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2 \cos \theta} r dr d\theta \\
&= 2 \int_0^{\frac{\pi}{3}} \left[\frac{r^2}{2} \right]_1^{2 \cos \theta} d\theta \\
&= \int_0^{\frac{\pi}{3}} 4 \cos^2 \theta - 1 d\theta \\
&= \int_0^{\frac{\pi}{3}} 4 \frac{\cos 2\theta + 1}{2} - 1 d\theta \\
&= \int_0^{\frac{\pi}{3}} 2 \cos 2\theta - 1 d\theta \\
&= \left[-\sin(2\theta) - \theta \right]_0^{\frac{\pi}{3}} \\
&= \sin\left(\frac{2\pi}{3} - \frac{\pi}{3} + \sin(0)\right) \\
&= \frac{\sqrt{3}}{2} - \frac{\pi}{3}
\end{aligned}$$

Exercise 21

Use polar coordinates to find the volume below the plane $2x + y + z = 4$ and above the disk $x^2 + y^2 \leq 1$.

$$\begin{aligned}
 x(x, y) &= 4 - 2x - y \\
 \iint_D 4 - 2x - y \, dA &= \int_0^{2\pi} \int_0^1 4 - 2(r \cos \theta) - (r \sin \theta)r \, dr \, d\theta \\
 &= \int_0^{2\pi} \left[2r - \frac{2r^2 \sin \theta}{3} + \frac{r^2 \cos \theta}{3} \right]_0^1 \, d\theta \\
 &= \int_0^{2\pi} 2 - \frac{2}{3} \sin \theta + \frac{1}{3} \cos \theta \, d\theta \\
 &= \left[2\theta + \frac{2}{3} \cos \theta - \frac{1}{3} \sin \theta \right]_0^{2\pi} \\
 &= 4\pi + \frac{2}{3} \cos 2\pi - \frac{1}{3} \sin 2\pi - 0 - \frac{2}{3} \cos 0 + \frac{1}{3} \sin 0 \\
 &= 4\pi
 \end{aligned}$$

Exercise 29

Evaluate the iterated integral by converting it to polar coordinates.

$$\begin{aligned}
 \int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx &= \int_0^{\frac{\pi}{2}} \int_0^2 e^{(-1)r^2} r \, dr \, d\theta \\
 &= \int_0^{\frac{\pi}{2}} \left[-\frac{e^{-r^2}}{2} \right]_0^2 \, d\theta \\
 &= \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{e^{-4}}{2} \, d\theta \\
 &= \frac{1}{2} \left[\theta - \frac{\theta}{e^4} \right]_0^{\frac{\pi}{2}} \\
 &= \frac{\pi}{4} - \frac{\pi}{4e^4}
 \end{aligned}$$

Exercise 31

Evaluate the iterated integral by converting it to polar coordinates.

$$\begin{aligned}\int_0^{\frac{1}{2}} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 \, dx \, dy &= \int_0^{\frac{\pi}{6}} \int_0^1 (r \cos \theta)(r \sin \theta)^2 r \, dr \, d\theta \\ &= \int_0^{\frac{\pi}{6}} \int_0^1 r^4 \cos \theta \sin^2 \theta \, dr \, d\theta \\ &= \int_0^{\frac{\pi}{6}} \left[\frac{r^5 \cos \theta \sin^2 \theta}{5} \right]_0^1 d\theta \\ &= \frac{1}{5} \int_0^{\frac{\pi}{6}} \cos \theta \sin^2 \theta \, d\theta \\ &= \frac{1}{5} \left[\frac{\sin^3 \theta}{3} \right]_0^{\frac{\pi}{6}} \\ &= \frac{1}{15} \left(\sin^3 \left(\frac{\pi}{6} \right) - \sin^3(0) \right) \\ &= \frac{1}{15} \left(\frac{1^3}{2^3} - 0 \right) \\ &= \frac{1}{120}\end{aligned}$$

Section 15.4

Exercise 7

Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ .

$$D \text{ is bounded by } y = 1 - x^2 \text{ and } y = 0; \quad \rho(x, y) = ky$$

$$\begin{aligned}0 &= 1 - x^2 \\ x &= \pm 1 \\ \text{mass} &= \iint_D \rho(x, y) \, dA \\ &= \int_{-1}^1 \int_0^{1-x^2} ky \, dy \, dx \\ &= 2 \int_0^1 \left[\frac{ky^2}{2} \right]_0^{1-x^2} dx \\ &= k \int_0^1 1 - 2x^2 + x^4 \, dx \\ &= k \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1 \\ &= k \left(1 - \frac{2}{3} + \frac{1}{5} \right) \\ &= \frac{8k}{15}\end{aligned}$$

$$\begin{aligned}
x_{CoM} &= \frac{1}{mass} \iint_D x\rho(x, y) \, dA \\
&= \frac{15}{8k} \int_{-1}^1 \int_0^{1-x^2} xky \, dy \, dx \\
&= k \frac{15}{8k} \int_{-1}^1 \int_0^{1-x^2} xy \, dy \, dx \\
&= \frac{15}{8} \int_{-1}^1 \left[\frac{xy^2}{2} \right]_0^{1-x^2} dx \\
&= \frac{15}{16} \int_{-1}^1 x(1 - 2x^2 + x^4) \, dx \\
&= \frac{15}{16} \int_{-1}^1 x - 2x^3 + x^5 \, dx \\
&= \frac{15}{16} \left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right]_{-1}^1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
y_{CoM} &= \frac{1}{mass} \iint_D y\rho(x, y) \, dA \\
&= \frac{15}{8k} \int_{-1}^1 \int_0^{1-x^2} ky^2 \, dy \, dx \\
&= \frac{15}{8} \int_{-1}^1 \left[\frac{y^3}{3} \right]_0^{1-x^2} dy \\
&= \frac{5}{8} \int_{-1}^1 (1 - x^2)^3 \, dx \\
&= \frac{5}{8} \int_{-1}^1 1 - x^6 - 3x^2 + 3x^4 \, dx \\
&= \frac{5}{8} \left[x - \frac{x^7}{7} - x^3 + \frac{3x^5}{5} \right]_{-1}^1 \\
&= \frac{5}{8} \left(1 - \frac{1}{7} - 1 + \frac{3}{5} + 1 - \frac{1}{7} - 1 + \frac{3}{5} \right) \\
&= \frac{5}{8} \left(-\frac{2}{7} + \frac{6}{5} \right) \\
&= \frac{5}{8} \frac{32}{35} \\
&= \frac{4}{7}
\end{aligned}$$

Center of mass: $\langle 0, \frac{4}{7} \rangle$

Exercise 15

Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a is the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

Treat the origin as the right angle with the legs a extending along the x and y axes.

$$\rho(x, y) = x^2 + y^2$$

$$\begin{aligned}
mass &= \iint_D \rho(x, y) \, dA \\
&= \int_0^a \int_0^{a-y} x^2 + y^2 \, dx \, dy \\
&= \int_0^a \left[\frac{x^3}{3} + xy^2 \right]_0^{a-y} \, dy \\
&= \int_0^a \frac{(a-y)^3}{3} + y^2(a-y) \, dy \\
&= \int_0^a \frac{1}{3}(a^3 - 3a^2y + 3ay^2 - y^3) + \frac{1}{3}(3ay^2 - 3y^3) \, dy \\
&= \frac{1}{3} \int_0^a a^3 - 3a^2y + 6ay^2 - 4y^3 \, dy \\
&= \frac{1}{3} \left[a^3y - \frac{3a^2y^2}{2} + 2ay^3 - y^4 \right]_0^a \\
&= \frac{1}{3} \left(a^4 - \frac{3a^4}{2} + 2a^4 - a^4 \right) \\
&= \frac{a^4}{6}
\end{aligned}$$

$$\begin{aligned}
x_{CoM} = y_{CoM} &= \frac{1}{mass} \iint_D y\rho(x, y) \, dA \\
&= \frac{6}{a^4} \int_0^a \int_0^{a-y} y(x^2 + y^2) \, dx \, dy \\
&= \frac{6}{a^4} \int_0^a \left[\frac{x^3y}{3} + xy^3 \right]_0^{a-y} \, dy \\
&= \frac{6}{a^4} \int_0^a \frac{y}{3}(a^3 - 3a^2y + 3ay^2 - y^3) + y^3(a-y) \, dy \\
&= \frac{6}{a^4} \int_0^a \frac{a^3y}{3} - a^2y^2 + ay^3 - \frac{y^4}{3} + ay^3 - y^4 \, dy \\
&= \frac{6}{a^4} \int_0^a \frac{a^3y}{3} - a^2y^2 + 2ay^3 - \frac{4y^4}{3} \, dy \\
&= \frac{6}{a^4} \left[\frac{a^3y^2}{6} - \frac{a^2y^3}{3} + \frac{2ay^4}{4} - \frac{4y^5}{15} \right]_0^a \\
&= \frac{6}{a^4} \left[\frac{a^5}{6} - \frac{a^5}{3} + \frac{a^5}{2} - \frac{4a^5}{15} \right] \\
&= a - 2a + 3a - \frac{8a}{5} \\
&= \frac{2a}{5}
\end{aligned}$$

Center of mass: $\langle \frac{2a}{5}, \frac{2a}{5} \rangle$

Section 15.6

Exercise 11

Evaluate the triple integral.

$$\begin{aligned}\iiint_E \frac{z}{x^2 + z^2} dV &= \int_1^4 \int_y^4 \int_0^z \frac{z}{x^2 + z^2} dx dz dy \\ &= \int_1^4 \int_y^4 z \left[\frac{1}{z} \tan^{-1}\left(\frac{x}{z}\right) \right]_0^z dz dy \\ &= \int_1^4 \int_y^4 \frac{\pi}{4} dz dy \\ &= \int_1^4 \left[\frac{\pi z}{4} \right]_y^4 dy \\ &= \int_1^4 \pi - \frac{\pi y}{4} dy \\ &= \left[\pi y - \frac{\pi y^2}{8} \right]_1^4 \\ &= 4\pi - 2\pi - \left(\pi - \frac{\pi}{8} \right) \\ &= \frac{9\pi}{8}\end{aligned}$$

Exercise 13

Evaluate the triple integral $\iiint_E 6xy dV$ where E lies below the plane $z = 1 + x + y$ and above the region in the xy -plane bounded by the curves $y = \sqrt{x}$, $y = 0$, $x = 1$.

$$\begin{aligned}\iiint_E 6xy dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy dz dy dx \\ &= \int_0^1 \int_0^{\sqrt{x}} \left[6xyz \right]_0^{1+x+y} dy dx \\ &= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y) dy dx \\ &= \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3 \right]_0^{\sqrt{x}} dx \\ &= \int_0^1 3x^2 + 3x^3 + 2x^{\frac{5}{2}} dx \\ &= \left[x^3 + \frac{3x^4}{4} + \frac{4}{7}x^{\frac{4}{2}} \right]_0^1 \\ &= 1 + \frac{3}{4} + \frac{4}{7} \\ &= \frac{28}{28} + \frac{21}{28} + \frac{16}{28} \\ &= \frac{65}{28}\end{aligned}$$

Exercise 21

Use a triple integral to find the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

$$\begin{aligned} & \{(x, y, z) \in E \mid -1 \leq x \leq 1, x^2 \leq y \leq 1, 0 \leq z \leq 1 - y\} \\ \iiint_E dV &= \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz \, dy \, dx \\ &= \int_{-1}^1 \int_{x^2}^1 \left[z \right]_0^{1-y} dy \, dx \\ &= \int_{-1}^1 \int_{x^2}^1 1 - y \, dy \, dx \\ &= \int_{-1}^1 \left[y - \frac{y^2}{2} \right]_{x^2}^1 dx \\ &= \int_{-1}^1 1 - \frac{1}{2} - x^2 + \frac{x^4}{2} dx \\ &= \left[\frac{x^5}{10} - \frac{x^3}{3} + \frac{x}{2} \right]_{-1}^1 \\ &= \frac{1}{10} - \frac{1}{3} + \frac{1}{2} + \frac{1}{10} - \frac{1}{3} + \frac{1}{2} \\ &= \frac{1}{5} - \frac{2}{3} + 1 \\ &= \frac{8}{15} \end{aligned}$$

Exercise 33

The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.

$$\begin{aligned} \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx &= \int_0^1 \int_0^{1-y} \int_0^{y^2} dx \, dy \, dz \\ &= \int_0^1 \int_0^{1-z} \int_0^{y^2} dx \, dz \, dy \\ &= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} dy \, dx \, dz \\ &= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} dy \, dz \, dx \\ &= \int_0^1 \int_0^{y^2} \int_0^{1-y} dz \, dx \, dy \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech