Multivariable and Vector Calculus: Homework 7

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Section 14.7

Exercise 9

Find the local maximum and minimum values and saddle point(s) of the function.

$$f(x,y) = x^2 + y^4 + 2xy$$

$$\begin{aligned}
f_x &= 2x + 2y \\
f_y &= 4y^3 + 2x \\
f_x &= 0 = 2x + 2y \\
x &= -y \\
f_y &= 4y^3 + 2x = 0 \\
&\equiv 4(-x)^3 + 2x = 0 \\
0 &= -4x^3 + 2x \\
0 &= x(-2x^2 + 1) \\
x &= 0 \quad x = \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

Points of interest: $(0,0), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$$\begin{aligned}
f_{xx} &= 2 \\
f_{xy} &= 2 \\
f_{yy} &= 12y^2 \\
D(0,0) &= 2(0) - 4 = -4 \quad \text{Saddle Point} \\
f(0,0) &= 0 \\
D(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) &= 2(6) - 4 = 8 \quad \text{Relative Minimum} \\
f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) &= \frac{1}{2} + \frac{1}{4} - 1 = -\frac{1}{4} \\
D(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) &= 2(6) - 4 = 8 \quad \text{Relative Minimum} \\
f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) &= -\frac{1}{4}
\end{aligned}$$

Exercise 17

$$f(x,y) = xy + e^{-xy}$$

$$\begin{aligned} f_x &= y - ye^{xy} \\ f_y &= x - xe^{xy} \\ f_x &= 0 = y - ye^{xy} \\ 0 &= y(1 - e^{xy}) \\ e^{xy} &= 1 \\ xy &= 0 \\ y &= 0 \text{ or } x = 0 \end{aligned}$$

Points of interest: $(0,y),(x,0)$

$$\begin{aligned} f_{xx} &= y^2 e^{xy} \\ f_{xy} &= 1 + xye^{-xy} - e^{-xy} \\ f_{yy} &= x^2 e^{-xy} \\ D(0,y) &= D(x,0) = 0 \end{aligned}$$

Inconclusive. By inspection of graph, $(0,y),(x,0)$ are local minima $\forall x,y$.

$$f(0,y) = 1 \quad f(x,0) = 1$$

Exercise 35

Find the absolute maximum or minimum values of f on the set D.

$$f(x,y) = x^2 + 2y^2 - 2x - 4y + 1 \quad D = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3\}$$

$$\begin{aligned} f_x &= 2x - 2 \\ f_y &= 4y - 4 \\ f_x &= 0 = 2x - 2 \\ x &= 1 \\ f_y &= 0 = 4y - 4 \\ y &= 1 \end{aligned}$$

Points of interest: $(1,1),(x,0),(x,3),(0,y),(2,y)$

- $(0,0)$

$$f(1,1) = -2$$

- $(x,0)$

$$\begin{aligned} f(x,0) &= x^2 - 2x + 1 \\ f_x &= 2x - 2 = 0 \\ x &= 1 \\ f(1,0) = 0 \quad f(0,0) &= 1 \quad f(2,0) = 1 \end{aligned}$$

- $(x, 3)$

$$\begin{aligned} f(x,3) &= x^2 + 18 - 2x - 12 + 1 \\ f_x &= 2x - 2 = 0 \\ x &= 1 \\ f(1,3) &= 0 \quad f(0,3) = 7 \quad f(2,3) = 7 \end{aligned}$$

- $(0, y)$

$$\begin{aligned} f(0,y) &= 2y^2 - 4y + 1 \\ f_y &= 4y - 4 = 0 \\ y &= 1 \\ f(0,1) &= -1 \quad f(0,0) = 1 \quad f(0,3) = 7 \end{aligned}$$

- $(2, y)$

$$\begin{aligned} f(2,y) &= 4 + 2y^2 - 4 - 4y + 1 \\ f_y &= 4y - 4 = 0 \\ y &= 1 \\ f(2,1) &= -1 \quad f(2,0) = 1 \quad f(2,3) = 7 \end{aligned}$$

Absolute Minimum: -2 at (1,1)
Absolute Maximum: 7 at (2,3), and (0,3)

Exercise 47

Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

$$\begin{aligned} \sqrt{l^2 + w^2 + h^2} &= 2r \\ l^2 + w^2 + h^2 &= 4r^2 \\ h &= \sqrt{4r^2 - l^2 - w^2} \\ V &= lwh \\ V(l,w) &= lw\sqrt{4t^2 - l^2 - w^2} \\ V_l &= w\sqrt{4t^2 - l^2 - w^2} - \frac{l^2 w}{\sqrt{4t^2 - l^2 - w^2}} \\ 0 &= w\sqrt{4t^2 - l^2 - w^2} - \frac{l^2 w}{\sqrt{4t^2 - l^2 - w^2}} \\ &= \frac{4t^2 w - l^2 w - w^3}{\sqrt{4t^2 - l^2 - w^2}} - \frac{l^2 w}{\sqrt{4t^2 - l^2 - w^2}} \\ &= w(4t^2 - 2l^2 - w^2) \end{aligned}$$

$$
\begin{aligned}
w &= 0 \quad w = \pm\sqrt{4t^2 - 2l^2} \\
V_w &= l\sqrt{4t^2 - l^2 - w^2} - \frac{lw^2}{\sqrt{4t^2 - l^2 - w^2}} \\
0 &= \frac{4t^2 l - l^3 - w^2 l}{\sqrt{4t^2 - l^2 - w^2}} - \frac{lw^2}{\sqrt{4t^2 - l^2 - w^2}} \\
&= l(4t^2 - l^2 - 2w^2) \\
&= 4t^2 - l^2 - 2(4t^2 - 2l^2) \\
&= 4t^2 - l^2 - 8t^2 + 4l^2 \\
4t^2 &= 3l^2 \\
l &= \sqrt{\frac{4t^2}{3}} = \frac{2r}{\sqrt{3}} \\
w &= \frac{2r}{\sqrt{3}} \\
h &= \frac{2r}{\sqrt{3}} \\
\text{maximum volume} &= lwh = \frac{8r^3}{3\sqrt{3}}
\end{aligned}
$$

Exercise 53

A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

$$
\begin{aligned}
V &= lwh = 32000 \\
h &= \frac{32000}{lw} \\
SA &= 2wh + 2lh + lw \\
&= 2w\frac{32000}{lw} + 2l\frac{32000}{lw} + lw \\
&= \frac{64000}{l} + \frac{64000}{w} + lw \\
SA_l &= -\frac{64000}{l^2} + w = 0 \\
w &= \frac{64000}{l^2} \\
SA_w &= -\frac{64000}{w^2} + l = 0 \\
l &= \frac{64000}{w^2} \\
l &= 64000\frac{l^4}{64000^2} \\
l^3 &= 64000 \\
l &= 40 \\
w &= 40 \\
h &= \frac{32000}{40^2} = 20
\end{aligned}
$$

Section 15.1

Exercise 1

Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x, y) \mid 0 \le x \le 6, 0 \le y \le 4\}$$

Use a Riemann sum with $m = 3, n = 2$ and take the sample point to be the upper right corner of each square.

$$\begin{aligned} V &= 4f(2,2) + 4f(4,2) + 4f(6,2) + 4f(4,2) + 4f(4,4) + 4f(4,6) \\ &= 4(4 + 8 + 12 + 8 + 16 + 24) \\ &= 16 + 32 + 48 + 32 + 64 + 96 \\ &= 48 + 80 + 160 \\ &= 288 \end{aligned}$$

Section 15.2

Exercise 9

Evaluate the double integral.

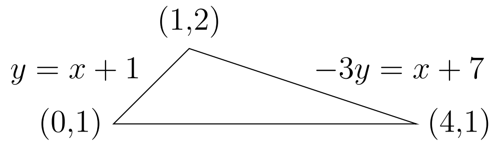
$$\iint_D \frac{y}{x^2+1} \, \mathrm{d}A \quad D = \{(x, y) \mid 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$$

$$\begin{aligned} \int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1} \, \mathrm{d}y \, \mathrm{d}x &= \int_0^4 \left[\frac{y^2}{2x^2+2} \right]_0^{\sqrt{x}} \mathrm{d}x \\ &= \frac{1}{2} \int_0^4 \frac{x}{x^2+1} \, \mathrm{d}x \\ u &= x^2 + 1 \\ \mathrm{d}u &= 2x \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^{4^2+1} \frac{x}{u} \frac{\mathrm{d}u}{2x} \\ &= \frac{1}{2} \int_0^{17} \frac{1}{2u} \, \mathrm{d}u \\ &= \frac{1}{4} \Big[\ln(u) \Big]_0^{17} \\ &= \frac{1}{4} (\ln(17) - \ln(1)) \\ &= \frac{\ln(17)}{4} \end{aligned}$$

Exercise 19

Evaluate the double integral.

$$\iint_D y^2 \, \mathrm{d}A \quad D \text{ is the triangular region with vertices } (0,1), (1,2), (4,1)$$



$$\begin{aligned}
\int_1^2 \int_{y-1}^{7-3y} y^2 \text{ d}x \text{ d}y &= \int_1^2 y^2 \Big[x\Big]_{y-1}^{7-3y} \text{ d}y \\
&= \int_1^2 y^2(7-3y-(y-1)) \text{ d}y \\
&= \int_1^2 8y^2 - 4y^3 \text{ d}y \\
&= \left[\frac{8y^3}{3} - \frac{4y^4}{4}\right]_1^2 \\
&= \frac{8(8)}{3} - 16 - (\frac{8}{3} - 1) \\
&= \frac{64}{3} - \frac{48}{3} - \frac{8}{3} + \frac{3}{3} \\
&= \frac{11}{3}
\end{aligned}$$

Exercise 23

Find the volume of the solid under the plane $3x+2y-z=0$ and above the region enclosed by the parabolas $y=x^2$ and $x=y^2$.

$$x = 3x + 2y$$

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} 3x+2y \text{ d}y \text{ d}x &= \int_0^1 \left[3xy + \frac{2y^2}{2}\right]_{x^2}^{\sqrt{x}} \\
&= \int_0^1 3x^{\frac{3}{2}} + x - (3x^3 + x^4) \text{ d}x \\
&= \int_0^1 3x^{\frac{3}{2}} - x^4 - 3x^3 + x \text{ d}x \\
&= \left[\frac{6}{5}x^{\frac{5}{2}} - \frac{x^5}{5} - \frac{3x^4}{4} + \frac{x^2}{2}\right]_0^1 \\
&= \frac{6}{5} - \frac{1}{5} - \frac{3}{4} + \frac{1}{2} - (0) \\
&= 1 - \frac{3}{4} + \frac{1}{2} \\
&= \frac{3}{4}
\end{aligned}$$

Exercise 29

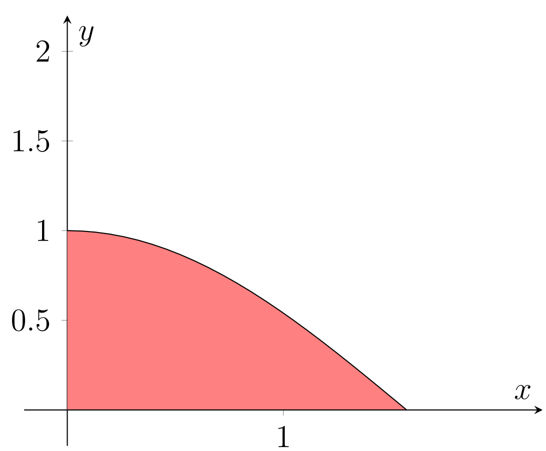
Find the volume of the solid enclosed by the cylinders $z = x^2, y = x^2$ and the planes $z = 0, y = 4$.

$$\begin{aligned}
\int_{-2}^{2}\int_{x^2}^{4} x^2 \; \mathrm{d}y \; \mathrm{d}x &= 2\int_{0}^{2}\int_{x^2}^{4} x^2 \; \mathrm{d}y \; \mathrm{d}x \\
&= 2\int_{0}^{2} x^2 \Big[y\Big]_{x^2}^{4} \; \mathrm{d}x \\
&= 2\int_{0}^{2} x^2(4 - x^2) \; \mathrm{d}x \\
&= 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 \\
&= 2(\frac{32}{3} - \frac{32}{5} - (0)) \\
&= 2(\frac{160}{3} - \frac{96}{15}) \\
&= 2(\frac{64}{15}) \\
&= \frac{128}{15}
\end{aligned}$$

Exercise 47

Sketch the region of integration and change the order of integration.

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{\cos(x)} f(x, y) \; \mathrm{d}y \; \mathrm{d}x$$

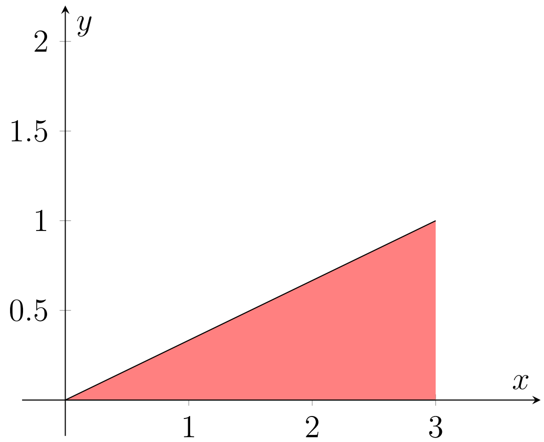


$$\int_{0}^{1}\int_{0}^{\cos^{-1}(y)} f(x, y) \; \mathrm{d}x \; \mathrm{d}y$$

Exercise 51

Evaluate the integral by reversing the order of integration.

$$\int_{0}^{1}\int_{3y}^{3} \mathrm{e}^{x^2} \; \mathrm{d}x \; \mathrm{d}y$$

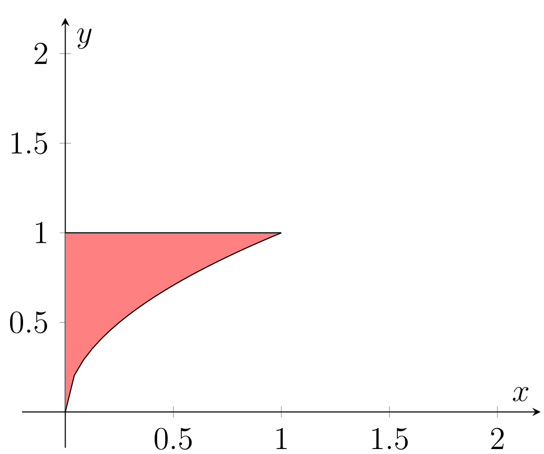


$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\, dy\, dx$$
$$= \int_0^3 e^{x^2} \Big[y\Big]_0^{\frac{x}{3}}\, dx$$
$$= \frac{1}{3}\int_0^3 x e^{x^2}\, dx$$
$$= \frac{1}{6}\int_0^3 2x e^{x^2}\, dx$$
$$= \frac{1}{6}\Big[e^{x^2}\Big]_0^3$$
$$= \frac{1}{6}(e^9 - e^0)$$
$$= \frac{e^9 - 1}{6}$$

Exercise 53

Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\, dy\, dx$$



$$
\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\ \mathrm{d}y\ \mathrm{d}x &= \int_0^1 \int_0^{y^2} \sqrt{y^3+1}\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_0^1 \sqrt{y^3+1}\Big[x\Big]_0^{y^2}\ \mathrm{d}y \\
&= \frac{1}{3}\int_0^1 3y^2\sqrt{y^3+1}\ \mathrm{d}y \\
&= \frac{1}{3}\left[\frac{2}{3}(y^3+1)^{\frac{3}{2}}\right]_0^1\ \mathrm{d}y \\
&= \frac{2}{9}(2^{\frac{3}{2}} - 1^{\frac{3}{2}}) \\
&= \frac{2(2\sqrt{2}-1)}{9} \\
&= \frac{4\sqrt{2}-2}{9}
\end{aligned}
$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech