

Multivariable and Vector Calculus: Homework 7

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August 2016 - December 2016

Section 14.7

Exercise 9

Find the local maximum and minimum values and saddle point(s) of the function.

$$f(x, y) = x^2 + y^4 + 2xy$$

$$f_x = 2x + 2y$$

$$f_y = 4y^3 + 2x$$

$$f_x = 0 = 2x + 2y$$

$$x = -y$$

$$f_y = 4y^3 + 2x = 0$$

$$\equiv 4(-x)^3 + 2x = 0$$

$$0 = -4x^3 + 2x$$

$$0 = x(-2x^2 + 1)$$

$$x = 0 \quad x = \pm \frac{1}{\sqrt{2}}$$

Points of interest: $(0, 0)$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$$f_{xx} = 2$$

$$f_{xy} = 2$$

$$f_{yy} = 12y^2$$

$$D(0, 0) = 2(0) - 4 = -4 \quad \text{Saddle Point}$$

$$f(0, 0) = 0$$

$$D(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 2(6) - 4 = 8 \quad \text{Relative Minimum}$$

$$f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{1}{4} - 1 = -\frac{1}{4}$$

$$D(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 2(6) - 4 = 8 \quad \text{Relative Minimum}$$

$$f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\frac{1}{4}$$

Exercise 17

$$f(x, y) = xy + e^{-xy}$$

$$f_x = y - ye^{xy}$$

$$f_y = x - xe^{xy}$$

$$f_x = 0 = y - ye^{xy}$$

$$0 = y(1 - e^{xy})$$

$$e^{xy} = 1$$

$$xy = 0$$

$$y = 0 \text{ or } x = 0$$

Points of interest: $(0, y), (x, 0)$

$$f_{xx} = y^2 e^{xy}$$

$$f_{xy} = 1 + xye^{-xy} - e^{-xy}$$

$$f_{yy} = x^2 e^{-xy}$$

$$D(0, y) = D(x, 0) = 0$$

Inconclusive. By inspection of graph, $(0, y), (x, 0)$ are local minima $\forall x, y$.

$$f(0, y) = 1 \quad f(x, 0) = 1$$

Exercise 35

Find the absolute maximum or minimum values of f on the set D .

$$f(x, y) = x^2 + 2y^2 - 2x - 4y + 1 \quad D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$$

$$f_x = 2x - 2$$

$$f_y = 4y - 4$$

$$f_x = 0 = 2x - 2$$

$$x = 1$$

$$f_y = 0 = 4y - 4$$

$$y = 1$$

Points of interest: $(1, 1), (x, 0), (x, 3), (0, y), (2, y)$

- $(0, 0)$

$$f(1, 1) = -2$$

- $(x, 0)$

$$f(x, 0) = x^2 - 2x + 1$$

$$f_x = 2x - 2 = 0$$

$$x = 1$$

$$f(1, 0) = 0 \quad f(0, 0) = 1 \quad f(2, 0) = 1$$

- $(x, 3)$

$$\begin{aligned}
 f(x, 3) &= x^2 + 18 - 2x - 12 + 1 \\
 f_x &= 2x - 2 = 0 \\
 x &= 1 \\
 f(1, 3) &= 0 \quad f(0, 3) = 7 \quad f(2, 3) = 7
 \end{aligned}$$

- $(0, y)$

$$\begin{aligned}
 f(0, y) &= 2y^2 - 4y + 1 \\
 f_y &= 4y - 4 = 0 \\
 y &= 1 \\
 f(0, 1) &= -1 \quad f(0, 0) = 1 \quad f(0, 3) = 7
 \end{aligned}$$

- $(2, y)$

$$\begin{aligned}
 f(2, y) &= 4 + 2y^2 - 4 - 4y + 1 \\
 f_y &= 4y - 4 = 0 \\
 y &= 1 \\
 f(2, 1) &= -1 \quad f(2, 0) = 1 \quad f(2, 3) = 7
 \end{aligned}$$

Absolute Minimum: -2 at $(1, 1)$

Absolute Maximum: 7 at $(2, 3)$, and $(0, 3)$

Exercise 47

Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r .

$$\begin{aligned}
 \sqrt{l^2 + w^2 + h^2} &= 2r \\
 l^2 + w^2 + h^2 &= 4r^2 \\
 h &= \sqrt{4r^2 - l^2 - w^2} \\
 V &= lwh \\
 V(l, w) &= lw\sqrt{4r^2 - l^2 - w^2} \\
 V_l &= w\sqrt{4r^2 - l^2 - w^2} - \frac{l^2w}{\sqrt{4r^2 - l^2 - w^2}} \\
 0 &= w\sqrt{4r^2 - l^2 - w^2} - \frac{l^2w}{\sqrt{4r^2 - l^2 - w^2}} \\
 &= \frac{4r^2w - l^2w - w^3}{\sqrt{4r^2 - l^2 - w^2}} - \frac{l^2w}{\sqrt{4r^2 - l^2 - w^2}} \\
 &= w(4r^2 - 2l^2 - w^2)
 \end{aligned}$$

$$\begin{aligned}
w = 0 \quad w &= \pm\sqrt{4t^2 - 2l^2} \\
V_w &= l\sqrt{4t^2 - l^2 - w^2} - \frac{lw^2}{\sqrt{4t^2 - l^2 - w^2}} \\
0 &= \frac{4t^2l - l^3 - w^2l}{\sqrt{4t^2 - l^2 - w^2}} - \frac{lw^2}{\sqrt{4t^2 - l^2 - w^2}} \\
&= l(4t^2 - l^2 - 2w^2) \\
&= 4t^2 - l^2 - 2(4t^2 - 2l^2) \\
&= 4t^2 - l^2 - 8t^2 + 4l^2 \\
4t^2 &= 3l^2 \\
l &= \sqrt{\frac{4t^2}{3}} = \frac{2r}{\sqrt{3}} \\
w &= \frac{2r}{\sqrt{3}} \\
h &= \frac{2r}{\sqrt{3}} \\
\text{maximum volume} &= lwh = \frac{8r^3}{3\sqrt{3}}
\end{aligned}$$

Exercise 53

A cardboard box without a lid is to have a volume of 32,000 cm³. Find the dimensions that minimize the amount of cardboard used.

$$\begin{aligned}
V &= lwh = 32000 \\
h &= \frac{32000}{lw} \\
SA &= 2wh + 2lh + lw \\
&= 2w\frac{32000}{lw} + 2l\frac{32000}{lw} + lw \\
&= \frac{64000}{l} + \frac{64000}{w} + lw \\
SA_l &= -\frac{64000}{l^2} + w = 0 \\
w &= \frac{64000}{l^2} \\
SA_w &= -\frac{64000}{w^2} + l = 0 \\
l &= \frac{64000}{w^2} \\
l &= 64000\frac{l^4}{64000^2} \\
l^3 &= 64000 \\
l &= 40 \\
w &= 40 \\
h &= \frac{32000}{40^2} = 20
\end{aligned}$$

Section 15.1

Exercise 1

Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x, y) \mid 0 \leq x \leq 6, 0 \leq y \leq 4\}$$

Use a Riemann sum with $m = 3, n = 2$ and take the sample point to be the upper right corner of each square.

$$\begin{aligned} V &= 4f(2, 2) + 4f(4, 2) + 4f(6, 2) + 4f(4, 2) + 4f(4, 4) + 4f(4, 6) \\ &= 4(4 + 8 + 12 + 8 + 16 + 24) \\ &= 16 + 32 + 48 + 32 + 64 + 96 \\ &= 48 + 80 + 160 \\ &= 288 \end{aligned}$$

Section 15.2

Exercise 9

Evaluate the double integral.

$$\iint_D \frac{y}{x^2 + 1} \, dA \quad D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq \sqrt{x}\}$$

$$\begin{aligned} \int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2 + 1} \, dy \, dx &= \int_0^4 \left[\frac{y^2}{2x^2 + 2} \right]_0^{\sqrt{x}} \, dx \\ &= \frac{1}{2} \int_0^4 \frac{x}{x^2 + 1} \, dx \\ u &= x^2 + 1 \\ du &= 2x \, dx \\ &= \frac{1}{2} \int_0^{4^2+1} \frac{x}{u} \frac{du}{2x} \\ &= \frac{1}{2} \int_0^{17} \frac{1}{2u} \, du \\ &= \frac{1}{4} \left[\ln(u) \right]_0^{17} \\ &= \frac{1}{4} (\ln(17) - \ln(1)) \\ &= \frac{\ln(17)}{4} \end{aligned}$$

Exercise 19

Evaluate the double integral.

$$\iint_D y^2 \, dA \quad D \text{ is the triangular region with vertices } (0, 1), (1, 2), (4, 1)$$

$$\begin{array}{c}
 (1,2) \\
 y = x + 1 \qquad \qquad \qquad -3y = x + 7 \\
 (0,1) \qquad \qquad \qquad \qquad \qquad (4,1)
 \end{array}$$

$$\begin{aligned}
 \int_1^2 \int_{y-1}^{7-3y} y^2 \, dx \, dy &= \int_1^2 y^2 \left[x \right]_{y-1}^{7-3y} dy \\
 &= \int_1^2 y^2 (7 - 3y - (y - 1)) \, dy \\
 &= \int_1^2 (8y^2 - 4y^3) \, dy \\
 &= \left[\frac{8y^3}{3} - \frac{4y^4}{4} \right]_1^2 \\
 &= \frac{8(8)}{3} - 16 - \left(\frac{8}{3} - 1 \right) \\
 &= \frac{64}{3} - \frac{48}{3} - \frac{8}{3} + \frac{3}{3} \\
 &= \frac{11}{3}
 \end{aligned}$$

Exercise 23

Find the volume of the solid under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

$$\begin{aligned}
 x &= 3x + 2y \\
 \int_0^1 \int_{x^2}^{\sqrt{x}} (3x + 2y) \, dy \, dx &= \int_0^1 \left[3xy + \frac{2y^2}{2} \right]_{x^2}^{\sqrt{x}} dx \\
 &= \int_0^1 (3x^{\frac{3}{2}} + x - (3x^3 + x^4)) \, dx \\
 &= \int_0^1 (3x^{\frac{3}{2}} - x^4 - 3x^3 + x) \, dx \\
 &= \left[\frac{6}{5}x^{\frac{5}{2}} - \frac{x^5}{5} - \frac{3x^4}{4} + \frac{x^2}{2} \right]_0^1 \\
 &= \frac{6}{5} - \frac{1}{5} - \frac{3}{4} + \frac{1}{2} - (0) \\
 &= 1 - \frac{3}{4} + \frac{1}{2} \\
 &= \frac{3}{4}
 \end{aligned}$$

Exercise 29

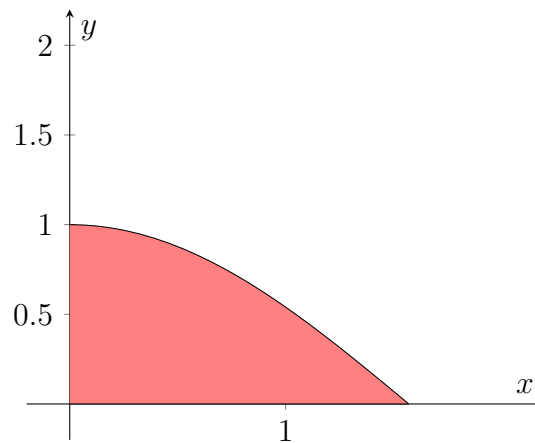
Find the volume of the solid enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$.

$$\begin{aligned}
 \int_{-2}^2 \int_{x^2}^4 x^2 \, dy \, dx &= 2 \int_0^2 \int_{x^2}^4 x^2 \, dy \, dx \\
 &= 2 \int_0^2 x^2 \left[y \right]_{x^2}^4 \, dx \\
 &= 2 \int_0^2 x^2(4 - x^2) \, dx \\
 &= 2 \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 \\
 &= 2 \left(\frac{32}{3} - \frac{32}{5} - (0) \right) \\
 &= 2 \left(\frac{160}{3} - \frac{96}{15} \right) \\
 &= 2 \left(\frac{64}{15} \right) \\
 &= \frac{128}{15}
 \end{aligned}$$

Exercise 47

Sketch the region of integration and change the order of integration.

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos(x)} f(x, y) \, dy \, dx$$

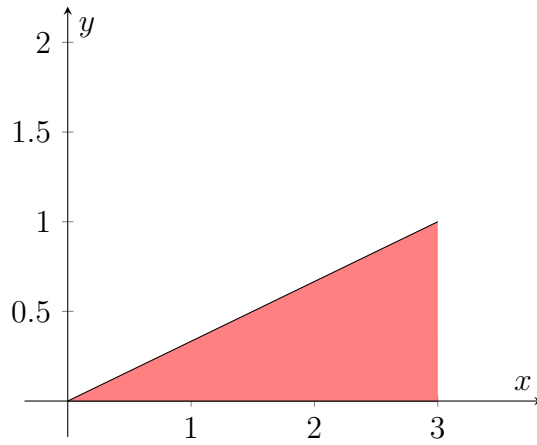


$$\int_0^1 \int_0^{\cos^{-1}(y)} f(x, y) \, dx \, dy$$

Exercise 51

Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy$$

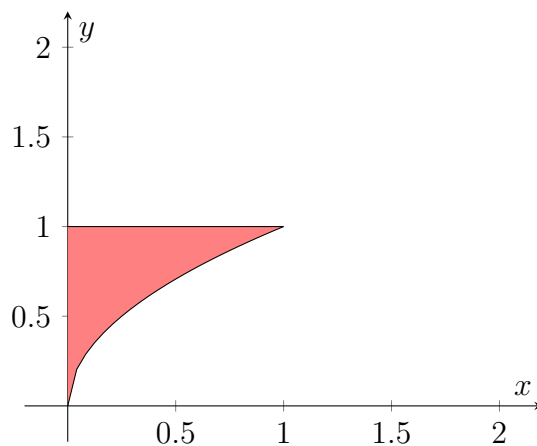


$$\begin{aligned}
 \int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy &= \int_0^3 \int_0^{\frac{x}{3}} e^{x^2} \, dy \, dx \\
 &= \int_0^3 e^{x^2} \left[y \right]_0^{\frac{x}{3}} \, dx \\
 &= \frac{1}{3} \int_0^3 x e^{x^2} \, dx \\
 &= \frac{1}{6} \int_0^3 2x e^{x^2} \, dx \\
 &= \frac{1}{6} \left[e^{x^2} \right]_0^3 \\
 &= \frac{1}{6} (e^9 - e^0) \\
 &= \frac{e^9 - 1}{6}
 \end{aligned}$$

Exercise 53

Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1} \, dy \, dx$$



$$\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1} \, dy \, dx &= \int_0^1 \int_0^{y^2} \sqrt{y^3 + 1} \, dx \, dy \\
&= \int_0^1 \sqrt{y^3 + 1} \left[x \right]_0^{y^2} \, dy \\
&= \frac{1}{3} \int_0^1 3y^2 \sqrt{y^3 + 1} \, dy \\
&= \frac{1}{3} \left[\frac{2}{3} (y^3 + 1)^{\frac{3}{2}} \right]_0^1 \, dy \\
&= \frac{2}{9} (2^{\frac{3}{2}} - 1^{\frac{3}{2}}) \\
&= \frac{2(2\sqrt{2} - 1)}{9} \\
&= \frac{4\sqrt{2} - 2}{9}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech