Multivariable and Vector Calculus: Homework 5

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Section 14.2

Exercise 5

Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$$

$$\begin{aligned}\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2) &= (3^2)(2^3) - 4(2^2) \\ &= 72 - 16 = 56\end{aligned}$$

Exercise 13

Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} &= \frac{0}{0} \text{ Indeterminate Form} \\ -|x| \leq &\frac{xy}{\sqrt{x^2+y^2}} \leq |x| \\ \lim_{(x,y)\to(0,0)} -|x| &= \lim_{(x,y)\to(0,0)} |x| = 0 \\ \therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} &= 0\end{aligned}$$

Exercise 15

Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2\cos(y)}{x^2+y^4}$$

$$y = 0$$

$$\lim_{(x,0)\to(0,0)} \frac{x(0)^2\cos(0)}{x^2+(0)^4} = 0$$

$$x = y^2$$

$$\begin{aligned}
\lim_{(y^2,y)\to(0,0)} \frac{(y^2)y^2\cos(y)}{(y^2)^2+y^4} &= \lim_{(y^2,y)\to(0,0)} \frac{y^4\cos(y)}{2y^4} \\
&= \lim_{(y^2,y)\to(0,0)} \frac{\cos(y)}{2} \\
&= \frac{1}{2}
\end{aligned}$$

$$0 \neq \frac{1}{2}$$

Limit does not exist.

Exercise 17

Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} &= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2+1-1} \\
&= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2} \\
&= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1) \\
&= \sqrt{1}+1 \\
&= 2
\end{aligned}$$

Section 14.3

Exercise 19

Find the first partial derivatives of the function.

$$z = \ln(x+t^2)$$

$$\frac{\partial z}{\partial x} = \frac{1}{x+t^2}$$

$$\frac{\partial z}{\partial t} = \frac{2t}{x+t^2}$$

Exercise 29

Find the first partial derivatives of the function.

$$F(x,y) = \int_y^x \cos(\mathrm{e}^t)\ \mathrm{d}t$$

$$\begin{aligned}\frac{\partial F}{\partial x} &= \cos(\mathrm{e}^x)\\ \frac{\partial F}{\partial y} &= -\cos(\mathrm{e}^y)\end{aligned}$$

Exercise 47

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$x^2 + 2y^2 + 3z^2 = 1$$

$$\begin{aligned}2x\frac{\partial x}{\partial z} + 0 + 6z\frac{\partial z}{\partial x} &= 0\\ 6z\frac{\partial z}{\partial x} &= -2x\\ \frac{\partial z}{\partial x} &= -\frac{2x}{6z}\\ &= -\frac{x}{3z}\\ 0 + 4y\frac{\partial y}{\partial y} + 6z\frac{\partial z}{\partial y} &= 0\\ 6z\frac{\partial z}{\partial y} &= -4y\\ \frac{\partial z}{\partial y} &= -\frac{4y}{6z}\\ &= -\frac{2y}{3z}\end{aligned}$$

Exercise 49

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\mathrm{e}^z = xyz$$

$$\begin{aligned}\mathrm{e}^z\frac{\partial z}{\partial x} &= y(x\frac{\partial z}{\partial x} + z\frac{\partial x}{\partial x})\\ \mathrm{e}^z\frac{\partial z}{\partial x} - yx\frac{\partial z}{\partial x} &= yz\\ \frac{\partial z}{\partial x}(\mathrm{e}^z - yx) &= yz\\ \frac{\partial z}{\partial x} &= \frac{yz}{\mathrm{e}^z - yx}\\ \mathrm{e}^z\frac{\partial z}{\partial y} &= x(y\frac{\partial z}{\partial y} + z\frac{\partial y}{\partial y})\\ \mathrm{e}^z\frac{\partial z}{\partial y} - xy\frac{\partial z}{\partial y} &= xz\\ \frac{\partial z}{\partial y}(\mathrm{e}^z - xy) &= xz\\ \frac{\partial z}{\partial y} &= \frac{xz}{\mathrm{e}^z - xy}\end{aligned}$$

Exercise 83

The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\frac{\partial R}{\partial R_1}$.

$$\ln(R)\frac{\partial R}{\partial R_1} = \ln(R_1)\frac{\partial R_1}{\partial R_1} + 0 + 0$$
$$\frac{\partial R}{\partial R_1} = \frac{\ln(R_1)}{\ln(R)}$$

Exercise 95

The kinetic energy of a body with mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that:

$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$$

$$\frac{\partial K}{\partial m} = \frac{1}{2}v^2$$
$$\frac{\partial K}{\partial v} = mv$$
$$\frac{\partial^2 K}{\partial v^2} = m$$
$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = (\frac{1}{2}v^2)(m) = K$$

Section 14.5

Exercise 3

Use the Chain Rule to find $\frac{\mathrm{d}z}{\mathrm{d}t}$.

$$z = \sin(x)\cos(y), \quad x = \sqrt{t}, \quad y = \frac{1}{t}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\frac{\partial z}{\partial y} = -\sin(x)\sin(y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -t^{-2}$$
$$\frac{\partial z}{\partial x} = \cos(x)\cos(y)$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{2\sqrt{t}}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\sin(x)\sin(y)}{t^2} - \frac{\cos(x)\cos(y)}{\sqrt{t}}$$

Exercise 27

Use Equation 6 to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$y\cos(x) = x^2 + y^2$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \\
F(x, y) &= 0 = x^2 + y^2 - y\cos(x) \\
\frac{\partial F}{\partial x} &= 2x + y\sin(x) \\
\frac{\partial F}{\partial y} &= 2y - \cos(x) \\
-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} &= -\frac{2x + y\sin(x)}{2y - \cos(x)}
\end{aligned}$$

Exercise 29

Use Equation 6 to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\tan^{-1}(x^2y) = x + xy^2$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \\
F(x, y) &= 0 = x + xy^2 - \tan^{-1}(x^2y) \\
\frac{\partial F}{\partial x} &= 1 + y^2 - \frac{2xy}{1 + x^4y^2} \\
\frac{\partial F}{\partial y} &= 2xy - \frac{x^2}{1 + x^4y^2}
\end{aligned}$$

$$-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{1 + y^2 - \frac{2xy}{1+x^4y^2}}{2xy - \frac{x^2}{1+x^4y^2}}$$

Exercise 33

Use Equations 7 to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\mathrm{e}^z = xyz$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$
$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$
$$F(x, y, z) = 0 = xyz - \mathrm{e}^z$$
$$\frac{\partial F}{\partial x} = yz$$
$$\frac{\partial F}{\partial y} = xz$$
$$\frac{\partial F}{\partial z} = xy - \mathrm{e}^z$$
$$\frac{\partial z}{\partial x} = -\frac{yz}{xy - \mathrm{e}^z}$$
$$\frac{\partial z}{\partial y} = -\frac{xz}{xy - \mathrm{e}^z}$$

Exercise 41

The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation PV = 8.31T in Example 2 to find the rate of change of the volume when the pressure is 20kPa and the temperature is 320K.

$$\frac{\partial P}{\partial t} = 0.05$$
$$\frac{\partial T}{\partial t} = 0.15$$
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}P}\frac{\partial P}{\partial t} + \frac{\mathrm{d}V}{\mathrm{d}T}\frac{\partial T}{\partial t}$$
$$PV = 8.31T$$
$$V(P, T) = \frac{8.31T}{P}$$
$$\frac{\partial V}{\partial P} = -\frac{8.31T}{P^2}$$
$$\frac{\partial V}{\partial T} = \frac{8.31}{P}$$
$$\frac{\mathrm{d}V}{\mathrm{d}t}(T, P) = -\frac{8.31T}{P^2}(0.05) + \frac{8.31}{P}(0.15)$$
$$\frac{\mathrm{d}V}{\mathrm{d}t}(320, 20) = -\frac{(0.05)(8.31)(320)}{20^2} + \frac{(0.15)(8.31)}{20}$$
$$= 0.27\frac{liters}{s}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech