

Multivariable and Vector Calculus: Homework 5

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Section 14.2

Exercise 5

Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y) \rightarrow (3,2)} (x^2y^3 - 4y^2)$$

$$\begin{aligned}\lim_{(x,y) \rightarrow (3,2)} (x^2y^3 - 4y^2) &= (3^2)(2^3) - 4(2^2) \\ &= 72 - 16 = 56\end{aligned}$$

Exercise 13

Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = \frac{0}{0} \text{ Indeterminate Form}$$

$$-|x| \leq \frac{xy}{\sqrt{x^2 + y^2}} \leq |x|$$

$$\lim_{(x,y) \rightarrow (0,0)} -|x| = \lim_{(x,y) \rightarrow (0,0)} |x| = 0$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0$$

Exercise 15

Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2 \cos(y)}{x^2 + y^4}$$

$$\begin{aligned}
& y = 0 \\
\lim_{(x,0) \rightarrow (0,0)} \frac{x(0)^2 \cos(0)}{x^2 + (0)^4} &= 0 \\
& x = y^2 \\
\lim_{(y^2,y) \rightarrow (0,0)} \frac{(y^2)y^2 \cos(y)}{(y^2)^2 + y^4} &= \lim_{(y^2,y) \rightarrow (0,0)} \frac{y^4 \cos(y)}{2y^4} \\
&= \lim_{(y^2,y) \rightarrow (0,0)} \frac{\cos(y)}{2} \\
&= \frac{1}{2} \\
0 &\neq \frac{1}{2}
\end{aligned}$$

Limit does not exist.

Exercise 17

Find the limit, if it exists, or show that the limit does not exist.

$$\begin{aligned}
& \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \\
\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)(\sqrt{x^2 + y^2 + 1} + 1)}{x^2 + y^2 + 1 - 1} \\
&= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)(\sqrt{x^2 + y^2 + 1} + 1)}{x^2 + y^2} \\
&= \lim_{(x,y) \rightarrow (0,0)} \sqrt{x^2 + y^2 + 1} + 1 \\
&= \sqrt{1} + 1 \\
&= 2
\end{aligned}$$

Section 14.3

Exercise 19

Find the first partial derivatives of the function.

$$z = \ln(x + t^2)$$

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{x + t^2} \\
\frac{\partial z}{\partial t} &= \frac{2t}{x + t^2}
\end{aligned}$$

Exercise 29

Find the first partial derivatives of the function.

$$F(x, y) = \int_y^x \cos(e^t) dt$$

$$\frac{\partial F}{\partial x} = \cos(e^x)$$

$$\frac{\partial F}{\partial y} = -\cos(e^y)$$

Exercise 47

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$x^2 + 2y^2 + 3z^2 = 1$$

$$2x \frac{\partial x}{\partial z} + 0 + 6z \frac{\partial z}{\partial x} = 0$$

$$6z \frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial x} = -\frac{2x}{6z}$$

$$= -\frac{x}{3z}$$

$$0 + 4y \frac{\partial y}{\partial y} + 6z \frac{\partial z}{\partial y} = 0$$

$$6z \frac{\partial z}{\partial y} = -4y$$

$$\frac{\partial z}{\partial y} = -\frac{4y}{6z}$$

$$= -\frac{2y}{3z}$$

Exercise 49

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$e^z = xyz$$

$$e^z \frac{\partial z}{\partial x} = y(x \frac{\partial z}{\partial x} + z \frac{\partial x}{\partial x})$$

$$e^z \frac{\partial z}{\partial x} - yx \frac{\partial z}{\partial x} = yz$$

$$\frac{\partial z}{\partial x}(e^z - yx) = yz$$

$$\frac{\partial z}{\partial x} = \frac{yz}{e^z - yx}$$

$$e^z \frac{\partial z}{\partial y} = x(y \frac{\partial z}{\partial y} + z \frac{\partial y}{\partial y})$$

$$e^z \frac{\partial z}{\partial y} - xy \frac{\partial z}{\partial y} = xz$$

$$\frac{\partial z}{\partial y}(e^z - xy) = xz$$

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

Exercise 83

The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\frac{\partial R}{\partial R_1}$.

$$\begin{aligned}\ln(R) \frac{\partial R}{\partial R_1} &= \ln(R_1) \frac{\partial R_1}{\partial R_1} + 0 + 0 \\ \frac{\partial R}{\partial R_1} &= \frac{\ln(R_1)}{\ln(R)}\end{aligned}$$

Exercise 95

The kinetic energy of a body with mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that:

$$\begin{aligned}\frac{\partial K}{\partial m} \frac{\partial^2 K}{\partial v^2} &= K \\ \frac{\partial K}{\partial m} &= \frac{1}{2}v^2 \\ \frac{\partial K}{\partial v} &= mv \\ \frac{\partial^2 K}{\partial v^2} &= m \\ \frac{\partial K}{\partial m} \frac{\partial^2 K}{\partial v^2} &= \left(\frac{1}{2}v^2\right)(m) = K\end{aligned}$$

Section 14.5

Exercise 3

Use the Chain Rule to find $\frac{dz}{dt}$.

$$z = \sin(x) \cos(y), \quad x = \sqrt{t}, \quad y = \frac{1}{t}$$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial y} \frac{dy}{dt} + \frac{\partial z}{\partial x} \frac{dx}{dt} \\ \frac{\partial z}{\partial y} &= -\sin(x) \sin(y) \\ \frac{dy}{dt} &= -t^{-2} \\ \frac{\partial z}{\partial x} &= \cos(x) \cos(y) \\ \frac{dx}{dt} &= \frac{1}{2\sqrt{t}} \\ \frac{dz}{dt} &= \frac{\sin(x) \sin(y)}{t^2} - \frac{\cos(x) \cos(y)}{\sqrt{t}}\end{aligned}$$

Exercise 27

Use Equation 6 to find $\frac{dy}{dx}$.

$$y \cos(x) = x^2 + y^2$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$$F(x, y) = 0 = x^2 + y^2 - y \cos(x)$$

$$\frac{\partial F}{\partial x} = 2x + y \sin(x)$$

$$\frac{\partial F}{\partial y} = 2y - \cos(x)$$

$$-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{2x + y \sin(x)}{2y - \cos(x)}$$

Exercise 29

Use Equation 6 to find $\frac{dy}{dx}$.

$$\tan^{-1}(x^2y) = x + xy^2$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$$F(x, y) = 0 = x + xy^2 - \tan^{-1}(x^2y)$$

$$\frac{\partial F}{\partial x} = 1 + y^2 - \frac{2xy}{1+x^4y^2}$$

$$\frac{\partial F}{\partial y} = 2xy - \frac{x^2}{1+x^4y^2}$$

$$-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{1 + y^2 - \frac{2xy}{1+x^4y^2}}{2xy - \frac{x^2}{1+x^4y^2}}$$

Exercise 33

Use Equations 7 to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$e^z = xyz$$

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \\ \frac{\partial z}{\partial y} &= -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \\ F(x, y, z) &= 0 = xyz - e^z \\ \frac{\partial F}{\partial x} &= yz \\ \frac{\partial F}{\partial y} &= xz \\ \frac{\partial F}{\partial z} &= xy - e^z \\ \frac{\partial z}{\partial x} &= -\frac{yz}{xy - e^z} \\ \frac{\partial z}{\partial y} &= -\frac{xz}{xy - e^z}\end{aligned}$$

Exercise 41

The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation $PV = 8.31T$ in Example 2 to find the rate of change of the volume when the pressure is 20kPa and the temperature is 320K.

$$\begin{aligned}\frac{\partial P}{\partial t} &= 0.05 \\ \frac{\partial T}{\partial t} &= 0.15 \\ \frac{dV}{dt} &= \frac{dV}{dP} \frac{\partial P}{\partial t} + \frac{dV}{dT} \frac{\partial T}{\partial t} \\ PV &= 8.31T \\ V(P, T) &= \frac{8.31T}{P} \\ \frac{\partial V}{\partial P} &= -\frac{8.31T}{P^2} \\ \frac{\partial V}{\partial T} &= \frac{8.31}{P} \\ \frac{dV}{dt}(T, P) &= -\frac{8.31T}{P^2}(0.05) + \frac{8.31}{P}(0.15) \\ \frac{dV}{dt}(320, 20) &= -\frac{(0.05)(8.31)(320)}{20^2} + \frac{(0.15)(8.31)}{20} \\ &= -0.27 \frac{\text{liters}}{\text{s}}\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech