Multivariable and Vector Calculus: Homework 2

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August 2016 - December 2016

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Exercise 5

Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both a and b.

$$\begin{aligned}
\vec{a} &= \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{4}\hat{k} \\
\vec{b} &= \hat{i} + 2\hat{j} - 3\hat{k} \\
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & 2 & -3 \end{vmatrix} = \frac{-3}{2}\hat{i} + \frac{7}{4}\hat{j} + \frac{2}{3}\hat{k} \\
\vec{a} \times \vec{b} \cdot \vec{a} &= \frac{-3}{4} + \frac{7}{12} + \frac{2}{12} = 0 \\
\vec{a} \times \vec{b} \cdot \vec{b} &= \frac{-3}{2} + \frac{14}{4} - 2 = 0
\end{aligned}$$

Exercise 7

Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both a and b.

$$\begin{aligned}
\vec{a} &= \langle t, 1, \frac{1}{t} \rangle \\
\vec{b} &= \langle t^2, t^2, 1 \rangle \\
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & 1 & \frac{1}{t} \\ t^2 & t^2 & 1 \end{vmatrix} = \langle 1 - t, 0, t^3 - t^2 \rangle \\
\vec{a} \times \vec{b} \cdot \vec{a} &= t(1 - t) + 0 + \frac{t^3 - t^2}{t} \\
&= t - t^2 + t^2 - t = 0 \\
\vec{a} \times \vec{b} \cdot \vec{b} &= t^2(1 - t) + 0 + t^3 - t^2 = 0
\end{aligned}$$

Exercise 11

Find the vector, not with determinants, but by using properties of cross products.

$$\begin{aligned}
(\hat{j} - \hat{k}) \times (\hat{k} - \hat{i}) &= (\hat{j} - \hat{k}) \times \hat{k} + (\hat{j} - \hat{k}) \times -\hat{i} \\
&= (\hat{j} \times \hat{k}) + (\hat{j} \times -\hat{i}) + (-\hat{k} \times \hat{k}) + (-\hat{k} \times -\hat{i}) \\
&= \hat{i} + \hat{k} + 0 + \hat{j} = \hat{i} + \hat{j} + \hat{k}
\end{aligned}$$

Exercise 13

State whether each expression is meaningful:

1. $\vec{a} \cdot (\vec{b} \times \vec{c})$
   Scalar

2. $\vec{a} \times (\vec{b} \cdot \vec{c})$
   Nonsensical (vector cross scalar)

3. $\vec{a} \times (\vec{b} \times \vec{c})$
   Vector

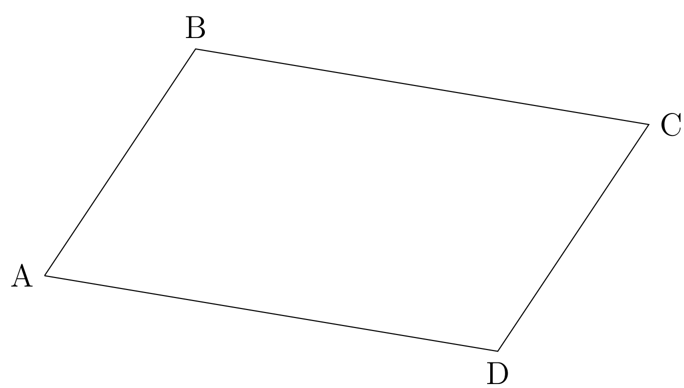
4. $\vec{a} \cdot (\vec{b} \cdot \vec{c})$
   Nonsensical (vector dot scalar)

5. $(\vec{a} \cdot \vec{b}) \times (\vec{c} \cdot \vec{d})$
   Nonsensical (scalar cross scalar)

6. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$
   Scalar

Exercise 27

Find the area of the parallelogram with vertices A(-3,0), B(-1,3), C(5,2), and D(3,-1).



$$\overrightarrow{AB} = \langle 2, 3 \rangle$$
$$\overrightarrow{AD} = \langle 6, -1 \rangle$$
$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ 6 & -1 & 0 \end{vmatrix} = \langle 0, 0, -20 \rangle$$

$$\text{area of ABCD} = |\langle 0, 0, -20 \rangle| = 20$$

Exercise 29

Find a nonzero vector orthogonal to the plane through the points P,Q, and R and find the area of △ PQR: P(1,0,1), Q(-2,1,3), R(4,2,5).

$$\overrightarrow{PQ} = \langle -3, 1, 2 \rangle$$
$$\overrightarrow{PR} = \langle 3, 2, 4 \rangle$$
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle 0, 18, 9 \rangle$$

$\langle 0, 2, 1 \rangle$ is a vector orthogonal to the plane and the area of △ PQR is:

$$\frac{1}{2}|\langle 0, 18, 9 \rangle| = \frac{\sqrt{405}}{2}$$

Exercise 33

Find the volume of the parallelepiped determine by the vectors:

$$\vec{a} = \langle 1, 2, 3 \rangle \quad \vec{b} = \langle -1, 1, 2 \rangle \quad \vec{c} = \langle 2, 1, 4 \rangle$$

$$\begin{aligned} V &= (\vec{a} \times \vec{b}) \cdot \vec{c} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix} \cdot \langle 2, 1, 4 \rangle \\ &= \langle 1, -5, 3 \rangle \cdot \langle 2, 1, 4 \rangle \\ &= 2 - 5 + 12 = 9 \end{aligned}$$

Exercise 37

Use the scalar triple product to verify that the vectors $\vec{u} = \hat{i} + 5\hat{j} - 2\hat{k}$, $\vec{v} = 3\hat{i} - \hat{j}$, and $w = 5\hat{i} + 9\hat{j} - 4\hat{k}$ are coplanar.

$$\begin{aligned} V &= (\vec{u} \times \vec{b}) \cdot \vec{w} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 5 & -2 \\ 3 & -1 & 0 \end{vmatrix} \cdot \langle 5, 9, -4 \rangle \\ &= \langle -2, -6, -16 \rangle \cdot \langle 5, 9, -4 \rangle \\ &= -10 - 54 + 64 = 0 \end{aligned}$$

Since the scalar triple product is 0, the vectors must be coplanar.

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Find a vector equation and parametric equations for the line through the point (1,0,6) and perpendicular to the plane $x + 3y + z = 5$.

$$\vec{n} = \langle 1, 3, 1 \rangle$$

$$\begin{aligned}\vec{r} &= \langle 1,0,6\rangle + \langle 1,3,1\rangle t \\ &= \langle 1+t, 3t, 6+t\rangle \\ &\equiv \begin{cases} x &= 1+t \\ y &= 3t \\ z &= 6+t \end{cases}\end{aligned}$$

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Find parametric equations and symmetric equations for the line through the points (-8,1,4) and (3,-2,4).

$$\vec{d} = \langle 11, -3, 0\rangle$$

$$\begin{aligned} l &= \begin{cases} x &= 3+11t \\ y &= -2-3t \\ z &= 4 \end{cases} \\ &\equiv \frac{x-3}{11} = \frac{y+2}{-3} \end{aligned}$$

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Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$$\begin{aligned} L_1 &= \begin{cases} x &= 3+2t \\ y &= 4-t \\ z &= 1+3t \end{cases} \\ L_2 &= \begin{cases} x &= 1+4s \\ y &= 3-2s\ \vec{n_1} \\ z &= 4+5s \end{cases} \qquad\qquad = \langle 2,-1,3\rangle \\ \vec{n_2} &= \langle 4,-2,5\rangle \\ 3+2t &= 1+4s \\ t &= 2s-1 \\ 4-t &= 3-2s \\ 4-(2s-1) &= 3-2s \\ 5-2s &= 3-2s \end{aligned}$$

No solution, therefore the lines are skew.

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Find an equation of the plane through the point (1,-1,-1) and parallel to the plane $5x-y-z=6$.

$$\begin{aligned} \vec{n} &= \langle 5,-1,-1\rangle \\ \Pi &\equiv 5(x-1) - 1(y+1) - 1(z+1) = 0 \\ &\equiv 5x-5-y-1-z-1 = 0 \\ &\equiv 5x-y-z = 7 \end{aligned}$$

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Find an equation of the plane through the points (0,1,1), (1,0,1), and (1,1,0).

$$\begin{aligned}
\vec{u} &= \langle 1, -1, 0 \rangle \\
\vec{v} &= \langle 1, 0, -1 \rangle \\
\vec{n} &= \vec{u} \times \vec{v} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle \\
\Pi &\equiv 1(x-0) + 1(y-1) + 1(z-1) = 0 \\
&\equiv x + y - 1 + z - 1 = 0 \\
&\equiv x + y + z = 2
\end{aligned}$$

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Find the equation of the plane that passes through the point (3,1,4) and contains the line of intersection of the planes $x + 2y + 3z = 1$ and $2x - y + z = -3$.
Let $z = 0$:

$$\begin{aligned}
x + 2y &= 1 \\
2x - y &= -3 \\
x &= 1 - 2y \\
2(1 - 2y) - y &= -3 \\
2 - 4y - y &= -3 \\
2 - 5y &= -3 \\
-5y &= -5 \\
y &= 1 \\
x &= -1
\end{aligned}$$

Let $y = 0$:

$$\begin{aligned}
x + 3z &= 1 \\
2x + z &= -3 \\
x &= 1 - 3z \\
2(1 - 3z) + z &= -3 \\
2 - 6z + z &= -3 \\
-5z &= -5 \\
z &= 1 \\
x &= -2
\end{aligned}$$

Points (-1,1,0) and (-2,0,1):

$$\begin{aligned}
\vec{u} &= (3,1,4) - (-1,1,0) = \langle 4,0,4\rangle \\
\vec{v} &= (3,1,3) - (-2,0,1) = \langle 5,1,2\rangle \\
\vec{n} &= \vec{u} \times \vec{v} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 4 \\ 5 & 1 & 2 \end{vmatrix} = \langle -4, 12, 4\rangle \\
\Pi &\equiv -4(x-(-1)) + 12(y-1) + 4(z-0) = 0 \\
&\equiv -4x - 4 + 12y - 12 + 4z = 0 \\
&\equiv -4x + 12y + 4z = 16 \\
&\equiv -x + 3y + z = 4 \\
&\equiv x - 3y - z = -4
\end{aligned}$$

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Use the formula to find the distance from the point to the given line. P = (4,1,-2)

$$l = \begin{cases} x & = 1 + t \\ y & = 3 - 2t \\ z & = 4 - 3t \end{cases}$$

$$\begin{aligned}
\vec{u} &= \langle 1, -2, -3\rangle \\
P_0 &= (1,3,4) \\
\vec{v} &= \overrightarrow{P_0P} = \langle 3, -2, -6\rangle \\
d &= \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|} \\
&= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 3 & -2 & -6 \end{vmatrix} \right\| \left(\frac{1}{|\langle 1, -2, -3\rangle|} \right) \\
&= \frac{|\langle 6, -3, 4\rangle|}{\sqrt{1+4+9}} \\
&= \frac{\sqrt{36+9+16}}{\sqrt{14}} \\
&= \sqrt{\frac{61}{14}}
\end{aligned}$$

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Find the distance from the point to the given plane. $P_1 = (1, -2, 4)$

$$3x + 2y + 6z = 5$$

$$
\begin{aligned}
D &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} \\
&= \frac{|3(1) + 2(-2) + 6(4) + (-5)|}{\sqrt{9+4+36}} \\
&= \frac{|3 - 4 + 24 - 5|}{\sqrt{49}} \\
&= \frac{18}{7}
\end{aligned}
$$

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Find the distance between the given parallel planes.

$$2x - 3y + z = 4 \quad 4x - 6y + 2z = 3$$

$$
\begin{aligned}
P_0 &= \langle 0, 0, 4 \rangle \\
D &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} \\
&= \frac{|4(0) - 6(0) + 2(4) - 3|}{\sqrt{16+36+4}} \\
&= \frac{|8 - 3|}{\sqrt{56}} \\
&= \frac{5}{4\sqrt{14}}
\end{aligned}
$$

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Show that the lines with symmetric equations $x = y = z$ and $x + 1 = \frac{y}{2} = \frac{z}{3}$ are skew, and find the distance between these lines.

$$
\begin{aligned}
l_1 &= \begin{cases} x &= t \\ y &= t \\ z &= t \end{cases} \\
l_2 &= \begin{cases} x &= -1 + s \\ y &= 2s \\ z &= 3s \end{cases} \\
t &= s - 1 \\
s - 1 &= 2s \\
s &= -1 \\
t &\stackrel{?}{=} 3s \\
s - 1 &= 3s \\
s &= \frac{-1}{2}
\end{aligned}
$$

Since the solutions do not match, the lines are skew.

$$\begin{aligned}
\vec{v_1} &= \langle 1, 1, 1 \rangle \\
\vec{v_2} &= \langle 1, 2, 3 \rangle \\
\vec{n} &= \vec{v_1} \times \vec{v_2} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 1, -2, 1 \rangle \\
P_0 &= (0, 0, 0) \\
\Pi &= 1x - 2y + 1z = 0 \\
P_1 &= (-1, 0, 0) \\
D &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|1(-1) + 0 + 0 + 0|}{\sqrt{1 + 4 + 1}} \\
&= \frac{1}{\sqrt{6}}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech