

Multivariable and Vector Calculus

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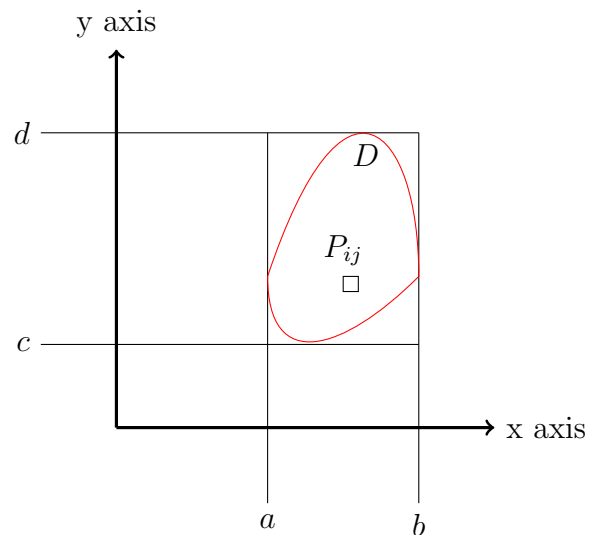
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Surface Integrals

If we have a surface S described by $r(u, v)$, $(u, v) \in D$, the area of the surface can be approximated as:

$$\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du dv$$

Find the mass of the surface S if the density of the surface is $f(x, y, z)$, $S : \overrightarrow{r(u, v)}$, $(u, v) \in D$.



$$D \subset R = [a, b] \times [c, d]$$

We divide $[a, b], [c, d]$ into nm equal parts to get R_{ij} .

$$mass \approx \sum_{i=1}^n \sum_{j=1}^m S_y f(P_{ij}) = \sum_{i=1}^n \sum_{j=1}^m \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| f(P_{ij}) \, du \, dv$$

If we take the limit as $n, m \rightarrow \infty$, this becomes the integral:

$$\iint_S f \, dS = \iint_D f \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| \, du \, dv$$

Example

Find $\iint_S z^2 \, dS$ given $S : z = y^2 + x^2, 0 \leq x \leq 1$.

$$\begin{aligned} S : r(R, \theta) &= \langle R^2, R \cos \theta, R \sin \theta \rangle \\ \left| \frac{\partial r}{\partial R} \times \frac{\partial r}{\partial \theta} \right| &= \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2R & \cos \theta & \sin \theta \\ 0 & -R \sin \theta & R \cos \theta \end{vmatrix} \right| \\ &= |\langle R, -2R^2 \cos \theta, -2R^2 \sin \theta \rangle| \\ &= \sqrt{R^2 + 4R^4 \cos^2 \theta + 4R^4 \sin^2 \theta} \\ &= R\sqrt{1 + 4R^2} \\ \iint_S z^2 \, dS &= \int_0^1 \int_0^{2\pi} R^2 \sin^2 \theta \, dS \\ &= \int_0^1 \int_0^{2\pi} R^2 \sin^2 \theta R \sqrt{1 + 4R^2} \, d\theta \, dR \\ &= \int_0^1 R^3 \sqrt{1 + 4R^2} \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{2\pi} \, dR \\ &= \pi \int_0^1 R^3 \sqrt{1 + 4R^2} \, dR \\ \text{Let : } u &= R^2 \quad dv = R\sqrt{1 + 4R^2} \, dR \\ du &= 2R \, dR \quad v = \frac{2}{3}(1 + 4R^2)^{\frac{3}{2}} \frac{1}{8} \\ &= \pi \left[R^2(1 + 4R^2)^{\frac{3}{2}} \frac{1}{2} - \int_0^1 \frac{1}{2}(1 + 4R^2)^{\frac{3}{2}} 2R \, dR \right] \\ &= \pi \left[R^2(1 + 4R^2)^{\frac{3}{2}} \frac{1}{2} - \frac{1}{6} \frac{2}{5} (1 + 4R^2)^{\frac{5}{2}} \frac{1}{8} \right] \end{aligned}$$

Example

Find $\iint_S y^2 \, dS$ given $S : x^2 + y^2 + z^2 = 1$ bounded above $z = \sqrt{x^2 + y^2}$.

$$\begin{aligned} r(\phi, \theta) &= \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle \\ \left| \frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} \right| &= \left\| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{array} \right\| \\ &= \sin \phi \\ \iint_S y^2 \, dS &= \int_0^{2\pi} \int_0^1 \sin^2 \phi \sin^2 \theta \sqrt{\sin^4 \theta + \cos^2 \phi \sin^2 \theta} \, d\phi \, d\theta \end{aligned}$$

Example

Find the surface area of sphere of radius R .

$$\begin{aligned} r(\phi, \theta) &= \langle R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi \rangle \\ \left| \frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} \right| &= \left\| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ R \cos \phi \cos \theta & R \cos \phi \sin \theta & -R \sin \phi \\ -R \sin \phi \sin \theta & R \sin \phi \cos \theta & 0 \end{array} \right\| \\ &= R^2 \sin \phi \\ \iint_S 1 \, dS &= \int_0^{2\pi} \int_0^1 R^2 \sin \phi \, d\phi \, d\theta \\ &= 4\pi R^2 \end{aligned}$$

Example

Suppose there is a function \vec{F} on S .

$$\begin{aligned} \iint_S \vec{F} \, dS &= \iint_S (\vec{F} \cdot \vec{r}) \, dS \\ &= \iint_S \vec{F} \cdot \frac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|} \, dS \\ &= \iint_D \vec{F} \cdot \frac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|} \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| \, du \, dv \\ &= \iint_D \vec{F} \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) \, du \, dv \end{aligned}$$

Example

Find $\iint_S \langle x, y, 5 \rangle dS$ given $S : x^2 + y^2 = 1, y = 0, x + y = 2$, picking an outwards normal vector to the surface.

$$\begin{aligned} r(\theta, y) &= \langle \cos \theta, y, \sin \theta \rangle \\ \frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial y} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin \theta & 0 & \cos \theta \\ 0 & 1 & 0 \end{vmatrix} \\ &= \langle -\cos \theta, 0, -\sin \theta \rangle \text{ reverse signs for outwards normal vector} \\ \iint_S \langle x, y, 5 \rangle dS &= \int_0^{2\pi} \int_0^{2-\cos \theta} \langle \cos \theta, y, 5 \rangle \cdot \langle \cos \theta, 0, \sin \theta \rangle dy d\theta \\ &= \int_0^{2\pi} \int_0^{2-\cos \theta} \cos^2 \theta + 5 \sin \theta dy d\theta \end{aligned}$$

Divergence Theorem

Let S be a oriented and closed surface.

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iiint_V \vec{\nabla} \cdot \vec{F} dV \\ \vec{\nabla} \cdot \vec{F} &= \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \end{aligned}$$

Example

$$\begin{aligned} \vec{\nabla} \cdot \langle x^y z, xy^2 z, xyz^2 \rangle &= 2xyz + 2xyz + 2xyz \\ \iint_S \langle x^y z, xy^2 z, xyz^2 \rangle dS &= \int_0^1 \int_0^1 \int_0^1 6xyz dz dy dx \end{aligned}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech