Multivariable and Vector Calculus

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Cylindrical Coordinates

$$(x, y, z) \to (r, \theta, z)$$

Cylindrical coordinates describe a point by using polar coordinates and an elevation.

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(\frac{y}{x})$$
$$z = z$$

Theorem

$$\iiint_V f(x, y, z) \text{ d}V = \iiint_C f(r\cos\theta, r\sin\theta, z)r \text{ d}z \text{ d}\theta \text{ d}r$$

Spherical Coordinates

The distance between the origin and a point $P(x, y, z)$ is ρ. We can project this point onto the xy-plane to point $P'(x, y, 0)$. We can compute the polar angle θ between P' and the x-axis. If we fix the distance ρ and let all other variables free, we get a sphere. If we fix θ and ρ, the resulting set of points yields a semicircle. If we take the angle ϕ between P and the y-axis, we get a system of coordinates (ρ, θ, ϕ) described by:

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}(\frac{y}{x})$$
$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \cos^{-1}\left(\frac{z}{\rho}\right)$$

Converting the other way:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\frac{\mathrm{d}(x, y, z)}{\mathrm{d}(\rho, \theta, \phi)} = \rho^2 \sin\phi$$

Example

$$(x, y, z) \to (\rho, \theta, \phi)$$
$$P(2, 2, 3) \equiv \left(\sqrt{17}, \frac{\pi}{4}, \cos^{-1}\left(\frac{3}{\sqrt{17}}\right)\right)$$

$\rho = 1$: Sphere
$\phi = \frac{\pi}{4}$: Cone
$\phi = \frac{\pi}{2}$: xy-plane
$\phi = \frac{3\pi}{4}$: Cone
$\phi = \pi$: Negative z-axis
$\phi = 0$: Positive z-axis

Theorem

$$\iiint_V f(x, y, z)\ \mathrm{d}V = \iiint_C f(r\cos\theta, r\sin\theta, z) r\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}z$$
$$= \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\ \mathrm{d}\rho\ \mathrm{d}\phi\ \mathrm{d}\theta$$

Average Value of a Function

$$A_v f = \frac{\iint_D f \text{ d}A}{\iint_D 1 \text{ d}A} = \frac{\iiint_V f \text{ d}V}{\iiint_V 1 \text{ d}V}$$

Center of Mass

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_D f \cdot x \text{ d}A}{\iint_D f \text{ d}A}$$

Moment of Inertia

$$J_y = \iint_D f \cdot x^2 \text{ d}A$$

Example

Find the center of mass of the solid bounded by the cone $r = 1, h = 1,\ f(x,y) = \sqrt{x^2 + y^2}$.

$$\begin{aligned}
\bar{x} &= \bar{y} = 0 \\
\bar{z} &= \frac{M_{xy}}{mass} \\
&= \frac{\iiint_V \sqrt{x^2+y^2} z \text{ d}V}{\iiint_V \sqrt{x^2+y^2} \text{ d}V} \\
&= \frac{\int_0^{2\pi} \int_0^1 \int_r^1 r^2 z \text{ d}z \text{ d}r \text{ d}\theta}{\int_0^{2\pi} \int_0^1 \int_r^1 r^2 \text{ d}z \text{ d}r \text{ d}\theta}
\end{aligned}$$

Note that this can also be represented by spherical coordinates:

$$= \frac{\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} \rho^4 \sin^3\phi \cos\phi \text{ d}\rho \text{ d}\phi \text{ d}\omega}{\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} \rho^3 \sin^3\phi \text{ d}\rho \text{ d}\phi \text{ d}\theta}$$

Example

Find the volume under $z = xy^2$ bounded by $z = 0$ and the triangle (1,0), (2,0), (4,0).

$$\begin{aligned}
V &= \iiint_V 1 \text{ d}V \\
&= \int_0^1 \int_{1+y}^{4-2y} \int_0^{x^2 y} 1 \text{ d}z \text{ d}x \text{ d}y
\end{aligned}$$

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