Multivariable and Vector Calculus

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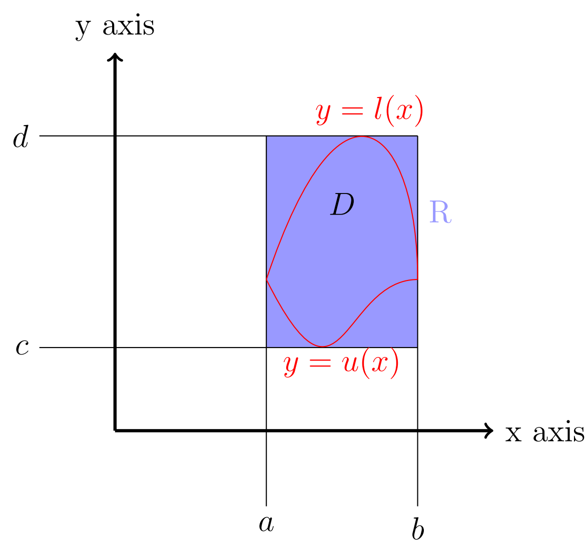
Double Integrals

Find the volume of a solid under $z = f(x, y)$ where:

$$(x, y) \in D \quad a \leq x \leq b \quad l(x) \leq y \leq u(x) \quad f(x, y) \geq 0 \text{ for } f(x, y) \in D$$



$$D \subset R = [a, b] \times [c, d]$$
$$V \approx |R| \cdot f(x_0, y_0), (x_0, y_0) \in D$$

This approximation is not very good. If we divide the region bounded by $[a, b]$ and $[c, d]$ into n and m equal parts to form R_{ij}:

$$V \approx \sum_{i=1}^{n} \sum_{i=1}^{m} |R_{ij}| f(x_{ij}, y_{ij})$$

This is the equivalent of a two dimensional Riemann sum.

Theorem

If $z = f(x, y)$ is continuous for $(x, y) \in D$, $y = l(x), y = u(x)$ are continuous for $a \leq x \leq b$ then there exists:

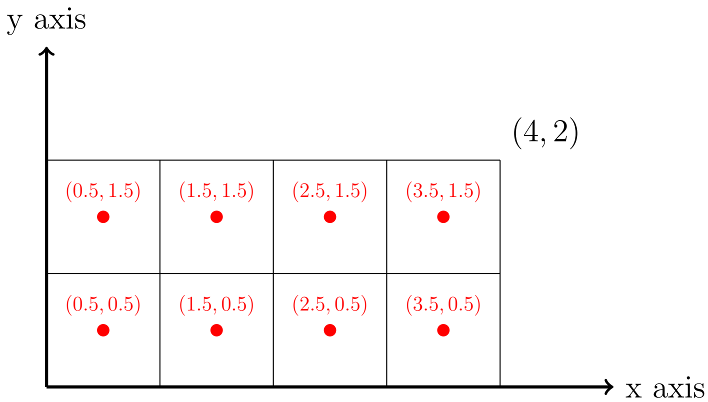
$$\lim_{n,m \to 0} \sum_{i=1}^{n} \sum_{i=1}^{m} |R_{ij}| f(x_{ij}, y_{ij})$$

which does not depend on the choice of points $(x_{ij}, y_{ij}) \in R_{ij}$. This limit is called the **double integral** of $f(x, y)$ on D and is equal to:

$$\iint_D f(x, y)\ \mathrm{d}A = \int_a^b \left[\int_{l(x)}^{u(x)} f(x, y)\ \mathrm{d}y \right]\ \mathrm{d}x$$

Example

Find the volume of a solid under $z = y^2, 0 \leq x \leq 4, 0 \leq y \leq 2$. Give the approximation of V when we divide $[a, b]$ into 4 parts and $[c, d]$ into 2 parts, picking points in the middle for our Riemann sum.



If we pick the center of each region (marked in red) as the approximation point:

$$\begin{aligned} V &\approx 1(\frac{1}{2})^2 + 1(\frac{1}{2})^2 + 1(\frac{1}{2})^2 + 1(\frac{1}{2})^2 + 1(\frac{3}{2})^2 + 1(\frac{3}{2})^2 + 1(\frac{3}{2})^2 + 1(\frac{3}{2})^2 \\ &\approx 4(\frac{3}{2})^2 + 4(\frac{1}{2})^2 \\ &\approx 10 \end{aligned}$$

To find the exact volume:

$$
\begin{aligned}
V &= \int_0^4 \int_0^2 y^2 \; \mathrm{d}y \; \mathrm{d}x \\
&= \int_0^4 \left[\frac{y^3}{3}\right]_0^2 \; \mathrm{d}x \\
&= \int_0^4 \frac{8}{3} \; \mathrm{d}x \\
&= \left[\frac{8}{3}x\right]_0^4 \\
&= \frac{32}{3}
\end{aligned}
$$

Example

Find the volume of the solid in the first octant bounded by $2x + y + z = 4$.

$$
\begin{aligned}
V &= \iint_D (4 - 2x - y) \; \mathrm{d}A \\
&= \int_0^2 \left[\int_0^{4-2x} (4 - 2x - y) \; \mathrm{d}y\right] \; \mathrm{d}x \\
&= \int_0^2 \left[(4 - 2x)y - \frac{y^2}{2}\right]_0^{4-2x} \; \mathrm{d}x \\
&= \int_0^2 (4 - 2x)^2 - \frac{(4 - 2x)^2}{2} \; \mathrm{d}x \\
&= \int_0^2 \frac{(4 - 2x)^2}{2} \; \mathrm{d}x \\
&= \left[\frac{1}{2}\frac{(4 - 2x)^3}{3}\frac{-1}{2}\right]_0^2 \\
&= \frac{4^3}{12} = \frac{16}{3}
\end{aligned}
$$

Example

Find the volume of the solid bounded by $x = z, y = x, x + y = 2, z = 0$.

$$\begin{aligned} V &= \iint_D x \, \mathrm{d}A \\ &= \int_0^1 \left[\int_x^{2-x} x \, \mathrm{d}y \right] \, \mathrm{d}x \end{aligned}$$

Example

Find the volume of the solid bounded by $z = 3y, z = 2 + y, y = x^2$.

$$\begin{aligned} V &= \iint_D 2 + y \, \mathrm{d}A - \iint_D 3y \, \mathrm{d}A \\ &= \iint_D 2 - 2y \, \mathrm{d}A \\ &= \int_{-1}^1 \left[\int_{x^2}^1 (2 - 2y) \, \mathrm{d}y \right] \, \mathrm{d}x \end{aligned}$$

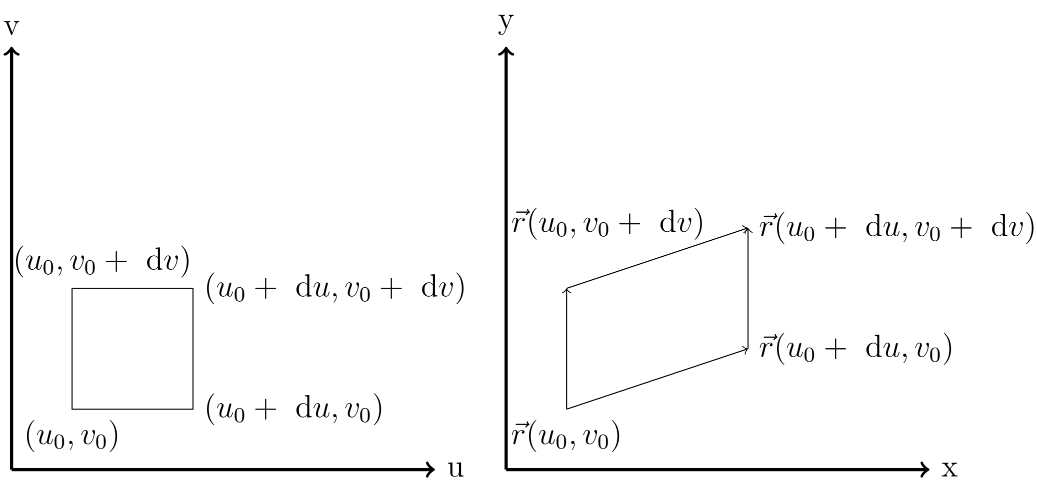
Example

Evaluate:

$$\begin{aligned} \int_0^8 \int_{\sqrt[3]{y}}^2 \mathrm{e}^{x^4} \, \mathrm{d}x \, \mathrm{d}y &= \int_0^2 \left[\int_0^{x^3} \mathrm{e}^{x^4} \, \mathrm{d}y \right] \, \mathrm{d}x \\ &= \int_0^2 \left[\mathrm{e}^{x^4} y \right]_0^{x^3} \, \mathrm{d}x \\ &= \int_0^2 \mathrm{e}^{x^4} x^3 \, \mathrm{d}x \\ &= \left[\frac{1}{4} \mathrm{e}^{x^4} \right]_0^2 \end{aligned}$$

Jacobian of Transformation

It is possible to simplify a problem by translating it between coordinate axes.



$$\begin{aligned}
\mathrm{d}A &\approx \left| \left(\frac{\vec{r}(u_0 + \mathrm{d}u, v_0) - \vec{r}(u_0, v_0)}{\mathrm{d}u} \right) \times \left(\frac{\vec{r}(u_0, v_0 + \mathrm{d}v) - \vec{r}(u_0, v_0)}{\mathrm{d}v} \right) \right| \mathrm{d}u \ \mathrm{d}v \\
&= \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| \mathrm{d}u \ \mathrm{d}v \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} \mathrm{d}u \ \mathrm{d}v \\
&= \left| \left\langle 0, 0, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right\rangle \right| \mathrm{d}u \ \mathrm{d}v \\
&= \frac{\partial(x,y)}{\partial(u,v)} \quad \text{Jacobian of transformation}
\end{aligned}$$

Theorem

If $\vec{r}$ is a one to one transformation of region D in the xy-plane and region $\triangle$ in the u, v coordinate plane, then

$$\iint_D f(x,y) \ \mathrm{d}A = \iint_\triangle f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)} \ \mathrm{d}u \ \mathrm{d}v$$

$C^T - x(u,v), y(u,v), \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial u}{\partial v}$ are all continuous. In the specific case of polar coordinates:

$$\iint_D f(x,y) \ \mathrm{d}A = \iint_\triangle f(r\cos\theta, r\sin\theta) r \ \mathrm{d}r \ \mathrm{d}\theta$$

$$R(x, y) = \left\langle \frac{r\cos\theta}{x}, \frac{r\sin\theta}{y} \right\rangle$$

Example

Find the volume of a solid under $z = \sqrt{x^2 + y^2}$ bounded by $1 \le x^2 + y^2 \le 4$.

$$\begin{aligned} V &= \iint_D \sqrt{x^2 + y^2}\ \mathrm{d}A \\ &= \int_0^{2\pi} \int_1^2 r^2\ \mathrm{d}r\ \mathrm{d}\theta \end{aligned}$$

And the computation becomes trivial.

Example

Find the volume of the solid under $z = 1 + 2x^2 + 2y^2$ bounded by $z = 9$ in the first octant.

$$\begin{aligned} V &= \iint_D 9 - (1 + 2x^2 + 2y^2)\ \mathrm{d}A \\ &= \int_0^{\frac{\pi}{2}} \int_0^2 (8 - 2r^2) r\ \mathrm{d}r\ \mathrm{d}\theta \end{aligned}$$

And the computation becomes trivial.

Example

Evaluate:

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\ \mathrm{d}y\ \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{8}{3}\cos^3\theta\ \mathrm{d}\theta \\
&= \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{2}} \\
&= \frac{8}{3}\frac{2}{3} \\
&= \frac{16}{9}
\end{aligned}$$

Example

Find the area inside $r = 1+\cos\theta$ outside $r = 3\cos\theta$.

$$\begin{aligned}
A &= 2\iint_D 1\ \mathrm{d}A \\
&= 2\left[\int_{\frac{\pi}{2}}^{\pi} \int_0^{1+\cos\theta} r\ \mathrm{d}r\ \mathrm{d}\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{3\cos\theta}^{1+\cos\theta} 1\ \mathrm{d}r\ \mathrm{d}\theta\right]
\end{aligned}$$

And the computation becomes trivial.

Example

Find the center of mass of $y = x^2, y = x+2$ given the mass density $\delta = x^2$.

$$\bar{x} = \frac{M_y}{mass} = \frac{\iint f \cdot x\ \mathrm{d}A}{\iint f\ \mathrm{d}A}$$

$$\bar{y} = \frac{M_x}{mass} = \frac{\iint f \cdot y\ \mathrm{d}A}{\iint f\ \mathrm{d}A}$$

$$
\begin{aligned}
\bar{x} &= \frac{\iint_D x^3 \ \mathrm{d}A}{\iint_D x^2 \ \mathrm{d}A} \\
&= \frac{\iint_D x^3 \ \mathrm{d}A}{\int_1^2 \int_{x^2}^{x+2} x^2 \ \mathrm{d}y \ \mathrm{d}x} \\
&= \frac{\iint_D x^3 \ \mathrm{d}A}{\int_1^2 \left[x^y \right]_{x^2}^{x+2} \ \mathrm{d}x} \\
&= \frac{\iint_D x^3 \ \mathrm{d}A}{\int_1^2 x^3 + 2x^2 - x^4 \ \mathrm{d}x}
\end{aligned}
$$

And the computation becomes trivial.

Example

Find the moment of inertia of $f(x, y) = x$ inside the square from (0,0) to (1,1).

$$
\begin{aligned}
J_x &= \iint_D f \cdot y^2 \ \mathrm{d}A \\
&= \iint_D xy^2 \ \mathrm{d}A \\
&= \int_0^1 \int_0^1 xy^2 \ \mathrm{d}y \ \mathrm{d}x \\
&= \int_0^1 \frac{x}{3} \ \mathrm{d}x \\
&= \frac{1}{6} \\
J_y &= \iint_D x \cdot x^2 \ \mathrm{d}A \\
&= \int_0^1 \int_0^1 x^3 \ \mathrm{d}y \ \mathrm{d}x \\
&= \int_0^1 x^3 \ \mathrm{d}x \\
&= \frac{1}{4}
\end{aligned}
$$

Extension to 3D

Find the mass of solid $V : a \leq x \leq b, l(x) \leq y \leq u(x), g(x,y) \leq z \leq h(x,y)$ if the mass density is $f(x,y,z)$.

- $V \subset B = [a,b] \times [c,d] \times [e,f]$
- divide $[a,b], [c,d], [e,f]$ into n, m, l parts to form B_{ijk} and pick point $(x_{ijk}, y_{ijk}, z_{ijk}) \in B_{ijk}$.

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} |B_{ijk}| \cdot f(x_{ijk}, y_{ijk}, z_{ijk})$$

Theorem: If $f(x,y,z)$ is continuous in V, $y = l(x), y = u(x)$ are continuous for $a \leq x \leq b$, and $g(x,y), h(x,y)$ are continuous for $(x,y) \in D$, then there exists a $\lim_{n,m,l \to \infty}$ which does not depend on the choice of points in B_{ijk}. This limit is called the triple integral of $f(x,y,z)$ on V and:

$$\iiint_V f(x,y,z)\ \mathrm{d}V = \int_a^b \left[\int_{l(x)}^{u(x)} \left[f(x,y,z)\ \mathrm{d}z \right] \mathrm{d}y \right] \mathrm{d}x$$

Mass:

$$\iiint_V f\ \mathrm{d}V$$

Volume:

$$\iint_V 1\ \mathrm{d}V$$

Example

Find the volume of the region bounded by $y = x^2, x = y^2, z = 0, z = y$.

$$\begin{aligned} V &= \iiint_V 1 \text{ d}V \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^y 1 \text{ d}z \text{ d}y \text{ d}x \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} \Big[z \Big]_0^y \text{ d}y \text{ d}x \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} y \text{ d}y \text{ d}x \\ &= \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} \text{ d}x \\ &= \int_0^1 \frac{x}{2} - \frac{x^4}{2} \text{ d}x \\ &= \left[\frac{x^2}{4} - \frac{x^5}{10} \right]_0^1 \\ &= \frac{1}{4} - \frac{1}{10} \end{aligned}$$

Example

Find the volume of the solid bounded by $x^2 + z^2 = 4, y = -1, y + z = 4$.

$$\begin{aligned} V &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} \text{ d}y \text{ d}z \text{ d}x \\ &= \int_{-2}^2 \int_{-1}^{4-z} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \text{ d}x \text{ d}y \text{ d}z \end{aligned}$$

Example

Find the mass of the solid if the mass is given by $\iiint_V z \text{ d}V$ and it is bounded by $y^2 + z^2 = 9, x = 0, y = 3x, z = 0$ in the first octant.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech