

Multivariable and Vector Calculus

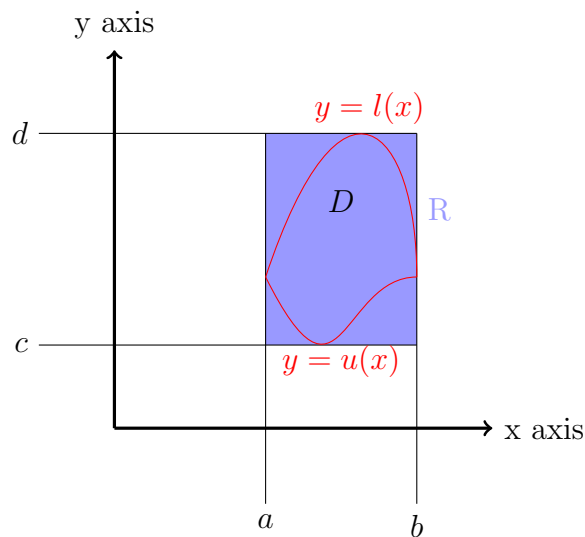
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Double Integrals

Find the volume of a solid under $z = f(x, y)$ where:

$$(x, y) \in D \quad a \leq x \leq b \quad l(x) \leq y \leq u(x) \quad f(x, y) \geq 0 \text{ for } f(x, y) \in D$$



$$D \subset R = [a, b] \times [c, d]$$

$$V \approx |R| \cdot f(x_0, y_0), (x_0, y_0) \in D$$

This approximation is not very good. If we divide the region bounded by $[a, b]$ and $[c, d]$ into n and m equal parts to form R_{ij} :

$$V \approx \sum_{i=1}^n \sum_{j=1}^m |R_{ij}| f(x_{ij}, y_{ij})$$

This is the equivalent of a two dimensional Riemann sum.

Theorem

If $z = f(x, y)$ is continuous for $(x, y) \in D$, $y = l(x), y = u(x)$ are continuous for $a \leq x \leq b$ then there exists:

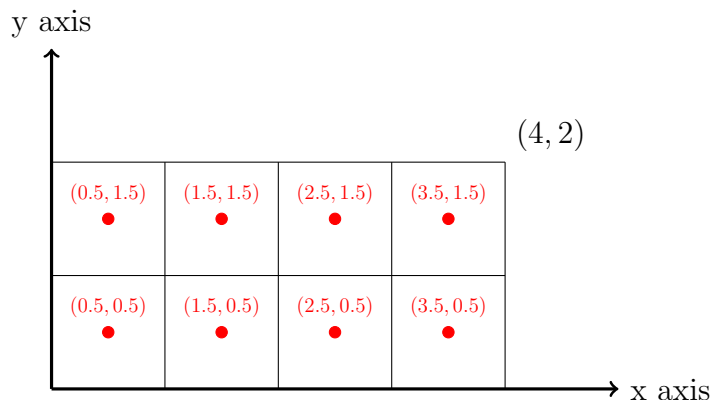
$$\lim_{n, m \rightarrow 0} \sum_{i=1}^n \sum_{j=1}^m |R_{ij}| f(x_{ij}, y_{ij})$$

which does not depend on the choice of points $(x_{ij}, y_{ij}) \in R_{ij}$. This limit is called the **double integral** of $f(x, y)$ on D and is equal to:

$$\iint_D f(x, y) \, dA = \int_a^b \left[\int_{l(x)}^{u(x)} f(x, y) \, dy \right] dx$$

Example

Find the volume of a solid under $z = y^2, 0 \leq x \leq 4, 0 \leq y \leq 2$. Give the approximation of V when we divide $[a, b]$ into 4 parts and $[c, d]$ into 2 parts, picking points in the middle for our Riemann sum.



If we pick the center of each region (marked in red) as the approximation point:

$$\begin{aligned} V &\approx 1\left(\frac{1}{2}\right)^2 + 1\left(\frac{1}{2}\right)^2 + 1\left(\frac{1}{2}\right)^2 + 1\left(\frac{1}{2}\right)^2 + 1\left(\frac{3}{2}\right)^2 + 1\left(\frac{3}{2}\right)^2 + 1\left(\frac{3}{2}\right)^2 + 1\left(\frac{3}{2}\right)^2 \\ &\approx 4\left(\frac{3}{2}\right)^2 + 4\left(\frac{1}{2}\right)^2 \\ &\approx 10 \end{aligned}$$

To find the exact volume:

$$\begin{aligned} V &= \int_0^4 \int_0^2 y^2 \, dy \, dx \\ &= \int_0^4 \left[\frac{y^3}{3} \right]_0^2 \, dx \\ &= \int_0^4 \frac{8}{3} \, dx \\ &= \left[\frac{8}{3}x \right]_0^4 \\ &= \frac{32}{3} \end{aligned}$$

Example

Find the volume of the solid in the first octant bounded by $2x + y + z = 4$.

$$\begin{aligned} V &= \iint_D (4 - 2x - y) \, dA \\ &= \int_0^2 \left[\int_0^{4-2x} (4 - 2x - y) \, dy \right] \, dx \\ &= \int_0^2 \left[(4 - 2x)y - \frac{y^2}{2} \right]_0^{4-2x} \, dx \\ &= \int_0^2 (4 - 2x)^2 - \frac{(4 - 2x)^2}{2} \, dx \\ &= \int_0^2 \frac{(4 - 2x)^2}{2} \, dx \\ &= \left[\frac{1}{2} \frac{(4 - 2x)^3}{3} - \frac{1}{2} \right]_0^2 \\ &= \frac{4^3}{12} = \frac{16}{3} \end{aligned}$$

Example

Find the volume of the solid bounded by $x = z, y = x, x + y = 2, z = 0$.

$$\begin{aligned} V &= \iint_D x \, dA \\ &= \int_0^1 \left[\int_x^{2-x} x \, dy \right] dx \end{aligned}$$

Example

Find the volume of the solid bounded by $z = 3y, z = 2 + y, y = x^2$.

$$\begin{aligned} V &= \iint_D 2 + y \, dA - \iint_D 3y \, dA \\ &= \iint_D 2 - 2y \, dA \\ &= \int_{-1}^1 \left[\int_{x^2}^1 (2 - 2y) \, dy \right] dx \end{aligned}$$

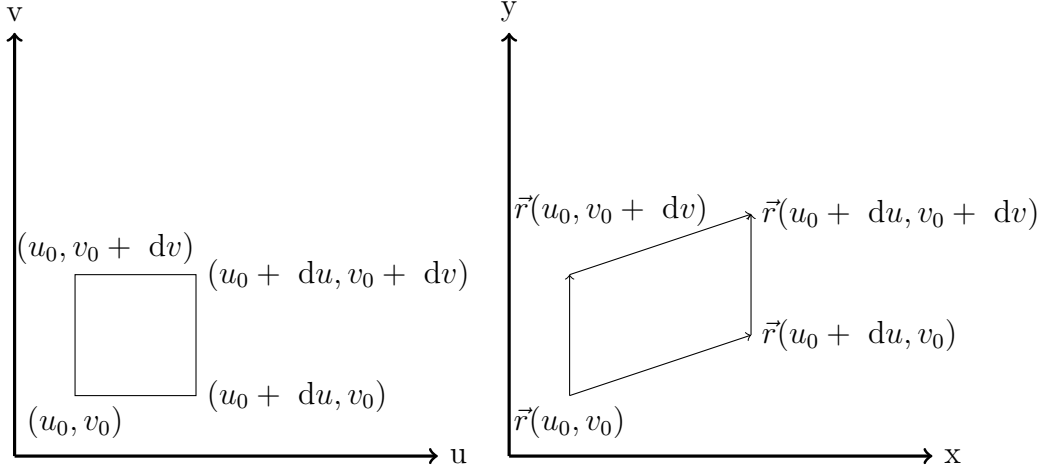
Example

Evaluate:

$$\begin{aligned} \int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4} \, dx \, dy &= \int_0^2 \left[\int_0^{x^3} e^{x^4} \, dy \right] dx \\ &= \int_0^2 \left[e^{x^4} y \right]_0^{x^3} dx \\ &= \int_0^2 e^{x^4} x^3 \, dx \\ &= \left[\frac{1}{4} e^{x^4} \right]_0^2 \end{aligned}$$

Jacobian of Transformation

It is possible to simplify a problem by translating it between coordinate axes.



$$\begin{aligned}
 dA &\approx \left| \left(\frac{\vec{r}(u_0 + du, v_0) - \vec{r}(u_0, v_0)}{du} \right) \times \left(\frac{\vec{r}(u_0, v_0 + dv) - \vec{r}(u_0, v_0)}{dv} \right) \right| du dv \\
 &= \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv \\
 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du dv \\
 &= \left| \left\langle 0, 0, \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right\rangle \right| du dv \\
 &= \frac{\partial(x, y)}{\partial(u, v)} \quad \text{Jacobian of transformation}
 \end{aligned}$$

Theorem

If \vec{r} is a one to one transformation of region D in the xy -plane and region Δ in the u, v coordinate plane, then

$$\iint_D f(x, y) dA = \iint_{\Delta} f(x(u, v), y(u, v)) \frac{\partial(x, y)}{\partial(u, v)} du dv$$

$C^T - x(u, v), y(u, v), \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ are all continuous. In the specific case of polar coordinates:

$$\iint_D f(x, y) dA = \iint_{\Delta} f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$R(x, y) = \left\langle \frac{r \cos \theta}{x}, \frac{r \sin \theta}{y} \right\rangle$$

Example

Find the volume of a solid under $z = \sqrt{x^2 + y^2}$ bounded by $1 \leq x^2 + y^2 \leq 4$.

$$\begin{aligned} V &= \iint_D \sqrt{x^2 + y^2} \, dA \\ &= \int_0^{2\pi} \int_1^2 r^2 \, dr \, d\theta \end{aligned}$$

And the computation becomes trivial.

Example

Find the volume of the solid under $z = 1 + 2x^2 + 2y^2$ bounded by $z = 9$ in the first octant.

$$\begin{aligned} V &= \iint_D 9 - (1 + 2x^2 + 2y^2) \, dA \\ &= \int_0^{\frac{\pi}{2}} \int_0^2 (8 - 2r^2)r \, dr \, d\theta \end{aligned}$$

And the computation becomes trivial.

Example

Evaluate:

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \, dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^{2\cos\theta} \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{8}{3} \cos^3\theta \, d\theta \\
&= \frac{8}{3} \left[\sin\theta - \frac{\sin^3\theta}{3} \right]_0^{\frac{\pi}{2}} \\
&= \frac{8}{3} \cdot \frac{2}{3} \\
&= \frac{16}{9}
\end{aligned}$$

ExampleFind the area inside $r = 1 + \cos\theta$ outside $r = 3\cos\theta$.

$$\begin{aligned}
A &= 2 \iint_D 1 \, dA \\
&= 2 \left[\int_{\frac{\pi}{2}}^{\pi} \int_0^{1+\cos\theta} r \, dr \, d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{3\cos\theta}^{1+\cos\theta} 1 \, dr \, d\theta \right]
\end{aligned}$$

And the computation becomes trivial.

ExampleFind the center of mass of $y = x^2, y = x + 2$ given the mass density $\delta = x^2$.

$$\begin{aligned}
\bar{x} &= \frac{M_y}{\text{mass}} = \frac{\iint f \cdot x \, dA}{\iint f \, dA} \\
\bar{y} &= \frac{M_x}{\text{mass}} = \frac{\iint f \cdot y \, dA}{\iint f \, dA}
\end{aligned}$$

$$\begin{aligned}
\bar{x} &= \frac{\iint_D x^3 \, dA}{\iint_D x^2 \, dA} \\
&= \frac{\iint_D x^3 \, dA}{\int_1^2 \int_{x^2}^{x+2} x^2 \, dy \, dx} \\
&= \frac{\iint_D x^3 \, dA}{\int_1^2 \left[x^y \right]_{x^2}^{x+2} dx} \\
&= \frac{\iint_D x^3 \, dA}{\int_1^2 x^3 + 2x^2 - x^4 \, dx}
\end{aligned}$$

And the computation becomes trivial.

Example

Find the moment of inertia of $f(x, y) = x$ inside the square from $(0,0)$ to $(1,1)$.

$$\begin{aligned}
J_x &= \iint_D f \cdot y^2 \, dA \\
&= \iint_D xy^2 \, dA \\
&= \int_0^1 \int_0^1 xy^2 \, dy \, dx \\
&= \int_0^1 \frac{x}{3} \, dx \\
&= \frac{1}{6}
\end{aligned}$$

$$\begin{aligned}
J_y &= \iint_D x \cdot x^2 \, dA \\
&= \int_0^1 \int_0^1 x^3 \, dy \, dx \\
&= \int_0^1 x^3 \, dx \\
&= \frac{1}{4}
\end{aligned}$$

Extension to 3D

Find the mass of solid $V : a \leq x \leq b, l(x) \leq y \leq u(x), g(x, y) \leq z \leq h(x, y)$ if the mass density is $f(x, y, z)$.

- $V \subset B = [a, b] \times [c, d] \times [e, f]$
- divide $[a, b], [c, d], [e, f]$ into n, m, l parts to form B_{ijk} and pick point $(x_{ijk}, y_{ijk}, z_{ijk}) \in B_{ijk}$.

$$\sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l |B_{ijk}| \cdot f(x_{ijk}, y_{ijk}, z_{ijk})$$

Theorem: If $f(x, y, z)$ is continuous in V , $y = l(x), y = u(x)$ are continuous for $a \leq x \leq b$, and $g(x, y), h(x, y)$ are continuous for $(x, y) \in D$, then there exists a $\lim_{n, m, l \rightarrow \infty}$ which does not depend on the choice of points in B_{ijk} . This limit is called the triple integral of $f(x, y, z)$ on V and:

$$\iiint_V f(x, y, z) \, dV = \int_a^b \left[\int_{l(x)}^{u(x)} \left[\int_{g(x, y)}^{h(x, y)} f(x, y, z) \, dz \right] dy \right] dx$$

Mass:

$$\iiint_V f \, dV$$

Volume:

$$\iiint_V 1 \, dV$$

Example

Find the volume of the region bounded by $y = x^2$, $x = y^2$, $z = 0$, $z = y$.

$$\begin{aligned} V &= \iiint_V 1 \, dV \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^y 1 \, dz \, dy \, dx \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} \left[z \right]_0^y \, dy \, dx \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} y \, dy \, dx \\ &= \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} \, dx \\ &= \int_0^1 \frac{x}{2} - \frac{x^4}{2} \, dx \\ &= \left[\frac{x^2}{4} - \frac{x^5}{10} \right]_0^1 \\ &= \frac{1}{4} - \frac{1}{10} \end{aligned}$$

Example

Find the volume of the solid bounded by $x^2 + z^2 = 4$, $y = -1$, $y + z = 4$.

$$\begin{aligned} V &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} dy \, dz \, dx \\ &= \int_{-2}^2 \int_{-1}^{4-z} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dx \, dy \, dz \end{aligned}$$

Example

Find the mass of the solid if the mass is given by $\iiint_V z \, dV$ and it is bounded by $y^2 + z^2 = 9$, $x = 0$, $y = 3x$, $z = 0$ in the first octant.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech