Multivariable and Vector Calculus

Alvin Lin

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Relative Maximum and Minimum

The function $z = f(x, y)$ has a relative local maximum at x_0, y_0 if and only if there exist $\epsilon > 0$, $f(x_0, y_0) \geq f(x, y)$ for all x, y and $dist((x, y), (x_0, y_0)) \leq \epsilon$. There exist **saddle points** where it may be a local maximum with respect to the derivative in one direction, but a local minimum with respective to the derivative in other directions.

Theorem

If $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ and

$$D(x_0, y_0) = f_{xx}f_{yy} - (f_{xy})^2 > 0$$

then f has a relative extremum. If $f_{xx} > 0$ then it is a minimum, if $f_{xx} < 0$ then it is a maximum. If $D(x_0, y_0) < 0$ we have a saddle point. If $D(x_0, y_0) = 0$, then the test is not conclusive.

Example

Find the relative maximum, minimum, and saddle points of $f(x, y) = 3xy - x^y - xy^2$.

$$\begin{aligned} f_x &= 3y - 2xy - y^2 = 0 \\ &\equiv y(3 - 2x - y) = 0 \\ y = 0 &\quad y = 3 - 2x \end{aligned}$$

$$\begin{aligned} f_y &= 3x - x^2 - 2xy = 0 \\ &\equiv x(3 - x - 2y) = 0 \end{aligned}$$

$$\begin{aligned}
\text{Plugging in } y &= 0 \\
x(3-x) &= 0 \\
x &= 0 \quad x = 3
\end{aligned}$$

$$\begin{aligned}
\text{Plugging in } y &= 3-2x \\
x(3-x-6+4x) &= 0 \\
x &= 0 \quad y = 3 \\
x &= 1 \quad y = 1
\end{aligned}$$

Points of interest: $(0,0),(3,0),(0,3),(1,1)$.

$$\begin{aligned}
&D(x,y) = (-2y)(-2x) - (3-2x-2y)^2 \\
&D(0,0) < 0 \text{ saddle} \\
&D(3,0) < 0 \text{ saddle} \\
&D(0,3) < 0 \text{ saddle} \\
&D(1,1) > 0 \text{ extremum, relative maximum}
\end{aligned}$$

Example

Find the distance between the point $P(1,2,3)$ and the plane $\Pi : x+y+z=7$.
In addition to the methods we have discussed earlier, we can also find this by minimizing the distance. We can pick a point $(x, y, 7-x-y)$ on the plane and find its distance to $(1,2,3)$.

$$dist^2 = F(x,y) = (x-1)^2 + (y-2)^2 + (-4+x+y)^2$$

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2(x-1) + 2(-4+x+y) \\
&= 2x-2-8+2x+2y \\
&= 4x+2y-10 = 0 \\
y &= 5-2x \\
\frac{\partial F}{\partial y} &= 2(y-2) + 2(-4+x+y) \\
&= 2y-4-8+2x+2y \\
&= 4y+2x-12 = 0
\end{aligned}$$

$$4(5-2x)+2x-12=0$$
$$20-8x+2x-12=0$$
$$8=6x$$
$$x=\frac{4}{3} \quad y=\frac{7}{3}$$

Example

Find the cheapest aquarium of volume 1 if the slate base costs 5 times the cost of glass.

$$xyz=1$$
$$C(x,y)=5xy+(2x\frac{1}{xy}+y\frac{1}{xy}2)=5xy+\frac{2}{y}+\frac{2}{x}$$
$$C_x=5y-\frac{2}{x^2}=0$$
$$y=\frac{2}{5x^2}$$
$$C_y=5x-\frac{2}{y^2}=0$$
$$5x-\frac{2}{(\frac{2}{5x^2})^2}=0$$
$$5x(1-\frac{5x^3}{2})=0$$
$$x=0 \quad x=\sqrt[3]{\frac{2}{5}}$$
$$y=\frac{2}{5(\frac{2}{5})^{\frac{2}{3}}}=\sqrt[3]{\frac{2}{5}}$$
$$z=\left(\frac{2}{5}\right)^{-\frac{2}{3}}$$

Example

Find the absolute maximum and minimum of $f(x,y)=x+y-xy$ within the region of the triangle bounded by $(0,0),(0,2),(4,0)$.

$$f_x=1-y=0$$
$$f_y=1-x=0$$

The only candidate point is $(1,1)$, and it is inside the triangular region defined. We also need to test the boundary points. Along the triangle's diagonal, there exists a relative maximum at the point $(\frac{3}{2}, \frac{5}{4})$.

$$f(1,1) = 1$$
$$f(0,0) = 0$$
$$f(4,0) = 4$$
$$f(0,2) = 2$$
$$f(\frac{3}{2}, \frac{5}{4}) = \frac{7}{8}$$

Thus the absolute minimum is at $(0,0)$ and the absolute maximum is at $(4,0)$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech