

Multivariable and Vector Calculus

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Relative Maximum and Minimum

The function $z = f(x, y)$ has a relative local maximum at x_0, y_0 if and only if there exist $\epsilon > 0$, $f(x_0, y_0) \geq f(x, y)$ for all x, y and $dist((x, y), (x_0, y_0)) \leq \epsilon$. There exist **saddle points** where it may be a local maximum with respect to the derivative in one direction, but a local minimum with respect to the derivative in other directions.

Theorem

If $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ and

$$D(x_0, y_0) = f_{xx}f_{yy} - (f_{xy})^2 > 0$$

then f has a relative extremum. If $f_{xx} > 0$ then it is a minimum, if $f_{xx} < 0$ then it is a maximum. If $D(x_0, y_0) < 0$ we have a saddle point. If $D(x_0, y_0) = 0$, then the test is not conclusive.

Example

Find the relative maximum, minimum, and saddle points of $f(x, y) = 3xy - x^2 - xy^2$.

$$\begin{aligned} f_x &= 3y - 2xy - y^2 = 0 \\ &\equiv y(3 - 2x - y) = 0 \\ y &= 0 \quad y = 3 - 2x \end{aligned}$$

$$\begin{aligned} f_y &= 3x - x^2 - 2xy = 0 \\ &\equiv x(3 - x - 2y) = 0 \end{aligned}$$

Plugging in $y = 0$

$$x(3 - x) = 0$$

$$x = 0 \quad x = 3$$

Plugging in $y = 3 - 2x$

$$x(3 - x - 6 + 4x) = 0$$

$$x = 0 \quad y = 3$$

$$x = 1 \quad y = 1$$

Points of interest: $(0, 0)$, $(3, 0)$, $(0, 3)$, $(1, 1)$.

$$D(x, y) = (-2y)(-2x) - (3 - 2x - 2y)^2$$

$$D(0, 0) < 0 \text{ saddle}$$

$$D(3, 0) < 0 \text{ saddle}$$

$$D(0, 3) < 0 \text{ saddle}$$

$$D(1, 1) > 0 \text{ extremum, relative maximum}$$

Example

Find the distance between the point $P(1, 2, 3)$ and the plane $\Pi : x + y + z = 7$.

In addition to the methods we have discussed earlier, we can also find this by minimizing the distance. We can pick a point $(x, y, 7 - x - y)$ on the plane and find its distance to $(1, 2, 3)$.

$$\text{dist}^2 = F(x, y) = (x - 1)^2 + (y - 2)^2 + (-4 + x + y)^2$$

$$\frac{\partial F}{\partial x} = 2(x - 1) + 2(-4 + x + y)$$

$$= 2x - 2 - 8 + 2x + 2y$$

$$= 4x + 2y - 10 = 0$$

$$y = 5 - 2x$$

$$\frac{\partial F}{\partial y} = 2(y - 2) + 2(-4 + x + y)$$

$$= 2y - 4 - 8 + 2x + 2y$$

$$= 4y + 2x - 12 = 0$$

$$\begin{aligned}
4(5 - 2x) + 2x - 12 &= 0 \\
20 - 8x + 2x - 12 &= 0 \\
8 &= 6x \\
x &= \frac{4}{3} \quad y = \frac{7}{3}
\end{aligned}$$

Example

Find the cheapest aquarium of volume 1 if the slate base costs 5 times the cost of glass.

$$\begin{aligned}
xyz &= 1 \\
C(x, y) &= 5xy + (2x\frac{1}{xy} + y\frac{1}{xy}2) = 5xy + \frac{2}{y} + \frac{2}{x} \\
C_x &= 5y - \frac{2}{x^2} = 0 \\
y &= \frac{2}{5x^2} \\
C_y &= 5x - \frac{2}{y^2} = 0 \\
5x - \frac{2}{(\frac{2}{5x^2})^2} &= 0 \\
5x(1 - \frac{5x^3}{2}) &= 0 \\
x = 0 \quad x &= \sqrt[3]{\frac{2}{5}} \\
y = \frac{2}{5(\frac{2}{5})^{\frac{2}{3}}} &= \sqrt[3]{\frac{2}{5}} \\
z &= \left(\frac{2}{5}\right)^{-\frac{2}{3}}
\end{aligned}$$

Example

Find the absolute maximum and minimum of $f(x, y) = x + y - xy$ within the region of the triangle bounded by $(0, 0), (0, 2), (4, 0)$.

$$\begin{aligned}
f_x &= 1 - y = 0 \\
f_y &= 1 - x = 0
\end{aligned}$$

The only candidate point is $(1, 1)$, and it is inside the triangular region defined. We also need to test the boundary points. Along the triangle's diagonal, there exists a relative maximum at the point $(\frac{3}{2}, \frac{5}{4})$.

$$f(1, 1) = 1$$

$$f(0, 0) = 0$$

$$f(4, 0) = 4$$

$$f(0, 2) = 2$$

$$f(\frac{3}{2}, \frac{5}{4}) = \frac{7}{8}$$

Thus the absolute minimum is at $(0, 0)$ and the absolute maximum is at $(4, 0)$.

You can find all my notes at <http://omgimanagerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanagerd.tech