

Multivariable and Vector Calculus

Alvin Lin

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Describing a line in \mathbb{R}^3

Consider a line l such that a point $P_0 \in l$ and $\vec{a} \parallel l$. To describe the line, for any point $P \in l$:

$$\overrightarrow{P_0P} \parallel \vec{a} \equiv \vec{r} - \vec{r}_0 = t\vec{a}$$

where \vec{r} is the vector from the origin to the point P and \vec{r}_0 is the vector from the origin to the point P_0 .

$$\begin{aligned}\vec{r} &= \vec{r}_0 + t\vec{a} \\ \langle x, y, z \rangle &= \langle x_0, y_0, z_0 \rangle + t\langle a_1, a_2, a_3 \rangle \\ x &= x_0 + ta_1 \\ y &= y_0 + ta_2 \\ z &= z_0 + ta_3 \\ \frac{x - x_0}{a_1} &= \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}\end{aligned}$$

Example

Line $l : P_0(1, 1, 2), P_1(2, 3, 5) \in l$

$$\begin{aligned}\vec{r} &= \langle 1, 1, 2 \rangle + t\langle 2 - 1, 3 - 1, 5 - 2 \rangle \\ &= \langle 1, 2, 3 \rangle \\ x &= 1 + t \\ y &= 1 + 2t \\ z &= 2 + 3t\end{aligned}$$

Example

There is a bug B traveling on l_1 :

$$l_1 = \begin{cases} x &= 2 + t \\ y &= 4 + 2t \\ z &= 1 + t \end{cases}$$

Another bug b is traveling on l_2 :

$$l_2 = \begin{cases} x &= 1 + t \\ y &= 4 + t \\ z &= 4 - t \end{cases}$$

Do their paths intersect at the same time?

$$\begin{aligned} x &= 2 + t = 1 + t \\ y &= 4 + 2t = 4 + t \\ z &= 1 + t = 4 - t \end{aligned}$$

Since there is no solution, they do not intersect at the same time. Is there a point of intersection between their paths at different times?

$$\begin{aligned} x &= 2 + t & &= 1 + \tau \\ t &= \tau - 1 \\ y &= 4 + 2t = 4 + \tau \\ 4 + 2(\tau - 1) &= 4 + \tau \\ 2 + 2\tau &= 4 + \tau \\ \tau &= 2 \\ t &= 1 \\ z &= 1 + t = 4 - \tau \\ 1 + 1 &\stackrel{?}{=} 4 - 2 \\ 2 &= 2 \end{aligned}$$

It is possible to solve for a τ and t , so the two paths do cross.

Example

In \mathbb{R}^2 we have the line $y = 2x + 3$, how do we convert this to \mathbb{R}^3 ? Two points on this line are $(0,3,0)$ and $(1,5,0)$:

$$l = \begin{cases} x &= 0 + t(1 - 0) = t \\ y &= 3 + t(5 - 3) = 3 + 2t \\ z &= 0 + t(0 - 0) = 0 \end{cases}$$

Example

Does the point $(1,2,-3)$ lie on the line:

$$\frac{x - 1}{3} = \frac{y + 2}{1} = \frac{z - 1}{-1}$$

With the point $(1,2,-3)$:

$$0 \neq 4 = 4$$

As a counterexample, $(1,-2,1)$ is a point that lies on the line.

Describing a plane in \mathbb{R}^3

Given a plane Π : $P_0 \in \Pi$, $\vec{n} \perp \Pi$:

$$P \in \Pi \equiv (\overrightarrow{P_0P} \perp \vec{n}) \equiv \overrightarrow{P_0P} \cdot \vec{n} = 0$$

where P is any point on plane Π . Plane Π is described by:

$$\begin{aligned} n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) &= 0 \\ n_1x + n_2y + n_3z &= 0 \end{aligned}$$

Example

Suppose the plane Π contains the points $A(1,1,1)$, $B(2,1,3)$, $C(1,4,4)$.

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 3 & 3 \end{vmatrix} = \langle 6, -3, 3 \rangle = \langle 2, 1, -1 \rangle$$

$$\begin{aligned} 2(x - 1) + 1(y - 1) - 1(z - 1) &= 0 \\ 2x + y - z &= 2 \end{aligned}$$

Example

Given a plane Π described by $2x + y - z = 5$, given a plane Π_1 parallel to Π containing the point $P_0(1,2,3)$.

$$\vec{n} = \langle 2, 1, -1 \rangle$$

$$\Pi_2 \equiv 2(x - 1) + 1(y - 2) - 1(z - 3) = 0 \equiv 2x + y - z = 1$$

Example

Given:

$$\Pi_1 \equiv x + y + z = 4$$

$$\Pi_2 \equiv 2x - y + 2z = 5$$

Find the line of intersection between the planes. Suppose we pick $z = 0$:

$$x + y = 4$$

$$2x - y = 5$$

Solving for this, we get the point $P_0(3,1,0)$. If we pick $x = 0$, we get the point $P_1(0,1,3)$. We can use these two points to get the line:

$$l = \begin{cases} x &= 3 + t(0 - 3) = 3 - 3t \\ y &= 1 + t(1 - 1) = 1 \\ z &= 0 + t(3 - 0) = 3t \end{cases}$$

What is the angle θ between Π_0, Π_1 ? We can compute the angle between the normal vectors of the planes.

$$\begin{aligned} \cos \angle(n_0, n_1) &= \frac{\vec{n}_0 \cdot \vec{n}_1}{|\vec{n}_0||\vec{n}_1|} \\ &= \frac{\langle 1, 1, 1 \rangle \cdot \langle 2, -1, 2 \rangle}{\sqrt{3}\sqrt{9}} \\ &= \frac{3}{3\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \\ \theta &= \cos^{-1} \frac{\sqrt{3}}{3} \end{aligned}$$

Example

Given two planes described by:

$$\Pi_1 : x + 2y + z = 4$$

$$\Pi_2 : x + 2x + z = 4$$

Find the distance between the planes. We can pick two arbitrary points on either plane and project the vector between them onto the normal vector. Let $P_1 = (4,0,0)$ and $P_2 = (6,0,0)$.

$$\begin{aligned} \text{dist}(\Pi_1, \Pi_2) &= \text{comp}_{\vec{n}} \overrightarrow{P_1 P_2} \\ &= \frac{\overrightarrow{P_1 P_2} \cdot \vec{n}}{|\vec{n}|} \\ &= \frac{\langle 2, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle}{\|\langle 1, 2, 1 \rangle\|} \\ &= \frac{2}{\sqrt{6}} \end{aligned}$$

Alternatively, we can use a point on Π_1 and find the equation of the line perpendicular to that point. We can find the intersection between that line and the Π_2 .

$$P_1 \in l, l \perp \Pi_1 : l = \begin{cases} x &= 4 + t \\ y &= 0 + 2t \\ z &= 0 + t \end{cases}$$

$\Pi_2 \cup l$ is the intersection between the perpendicular line and Π_2 .

$$\begin{aligned} x + 2y + z &= 6 \\ (4 + t) + 2(2t) + (t) &= 6 \\ 6t + 4 &= 6 \\ t &= \frac{1}{3} \end{aligned}$$

Using this t , we can get the point $(\frac{13}{3}, \frac{2}{3}, \frac{1}{3})$. We can then use the distance formula between this point and $(4,0,0)$ to get the same result.

Example

Extending the previous problem, find Π_3 such that the distance between the planes is 3. There exists a point $(d, 0, 0)$ on such a plane that satisfies this condition.

$$\begin{aligned} \text{dist}(\Pi_1, \Pi_3) &= \text{comp}_{\vec{n}} \overrightarrow{P_1 P_3} \\ 3 &= \frac{\overrightarrow{P_1 P_3} \cdot \vec{n}}{|\vec{n}|} \\ &= \frac{\langle d-4, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle}{|\langle 1, 2, 1 \rangle|} \\ &= \frac{d-4}{\sqrt{6}} \\ d &= 4 + 3\sqrt{6} \end{aligned}$$

Example

Find Π such that $(1, 2, 3) \in \Pi$:

$$\begin{aligned} \Pi \perp \Pi_1 &: x - y = 2 \\ \Pi \perp \Pi_2 &: 2x + z = 4 \end{aligned}$$

These two lines can describe two vectors parallel to Π . Therefore, their cross product is a vector perpendicular to Π , which is the normal of Π .

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \langle -1, -1, 2 \rangle$$

$$\Pi : -1(x-1) - 1(y-2) + 2(z-3) = 0$$

Example

Find Π :

$$\begin{aligned} l &\in \Pi \\ l &= \Pi_1 \cap \Pi_2 \\ \Pi_1 &: x - 2 = 1 \\ \Pi_2 &: 2x + y + 2 = 4 \end{aligned}$$

$$\Pi \perp \Pi_3 : x + 2y = 1$$

We can find two points in Π since they must satisfy the following equations:

$$\begin{aligned} x - z &= 1 \\ 2x + y + z &= 4 \end{aligned}$$

We can use $P_0 = (0, 5, -1)$ and $P_1 = (1, 2, 0)$. These two points can describe a vector parallel to Π . We can take the cross product between $\overrightarrow{P_0P_1}$ and $\Pi \perp \Pi_3$ to get the normal of Π .

$$\vec{n} = \overrightarrow{P_0P_1} \times \vec{n}_3 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 1 \\ 1 & 2 & 0 \end{vmatrix} = \langle -2, 1, 5 \rangle$$

$$\Pi : -2(x - 1) + 1(y - 2) + 5(z - 0) = 0$$

Example

Find line l given:

$$P(0, 1, 2) \in l$$

$$l \parallel \Pi : x + y + z = 1$$

$$l \perp l_1 = \begin{cases} x = 1 + t \\ y = 1 - t \\ z = 2t \end{cases}$$

The vector \vec{a} pointing in the direction of l_1 is perpendicular to both Π and l . The normal of Π is perpendicular to l . We can take $\vec{a} \times \vec{n}$ to get a vector \vec{a}_l describing the direction of l .

$$\vec{a}_l = \vec{a}_1 \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \langle -3, 1, 2 \rangle$$

$$l = \begin{cases} x = 0 + 3t \\ y = 1 + t \\ z = 2 + 2t \end{cases}$$

Example

Given the parallel lines:

$$l_1 = \begin{cases} x &= 1 + t \\ y &= 0 + 2t \\ z &= 0 + 4t \end{cases} \quad l_2 = \begin{cases} x &= 0 + t \\ y &= 1 + 2t \\ z &= 1 + 4t \end{cases}$$

Find the distance between the lines.

We can pick two arbitrary points on l_1 : $P_1(1,0,0)$ and $P_2(2,2,4)$, and one arbitrary point on l_2 : $P_3(0,1,1)$. The area of the parallelogram described by $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_1P_2}$ is:

$$\begin{aligned} \text{area} &= |\overrightarrow{P_1P_3} \times \overrightarrow{P_1P_2}| \\ &= |\overrightarrow{P_1P_3} \cdot \text{dist}| \\ \text{dist}(l_1, l_2) &= \frac{|\overrightarrow{P_1P_3} \times \overrightarrow{P_1P_2}|}{|\overrightarrow{P_1P_3}|} \\ &= \frac{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ -1 & 1 & 12 \end{vmatrix}}{|\langle 1, 2, 4 \rangle|} \\ &= \frac{|\langle -2, -5, 3 \rangle|}{|\langle 1, 2, 4 \rangle|} \\ &= \frac{\sqrt{38}}{\sqrt{21}} \end{aligned}$$

Example

Find the shortest distance between the skew lines:

$$l_1 = \begin{cases} x &= 1 + t \\ y &= 0 + 2t \\ z &= 0 + 4t \end{cases} \quad l_2 = \begin{cases} x &= 0 + s \\ y &= 1 + 2s \\ z &= 1 + s \end{cases}$$

If we take a vector \vec{a}_1 in the direction of l_1 and a vector \vec{a}_2 in the direction of l_2 , we can take their cross product to get a vector describing the normal of a plane Π parallel to both lines such that $l_1 \in \Pi$.

$$\vec{n} = \vec{a}_1 \times \vec{a}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ 1 & 2 & 1 \end{vmatrix} = \langle -6, 3, 0 \rangle \parallel \langle -2, 1, 0 \rangle$$

The plane Π can be described by:

$$2(x - 1) - 1(y - 0) + 0(z - 0) = 0$$

$$2x - y = 2$$

This reduces the problem to simply the distance between l_2 and Π .

$$\begin{aligned} \text{dist}(l_1, l_2) &= \text{comp}_{\langle 2, -1, 0 \rangle} \langle -1, 1, 1 \rangle \\ &= \frac{\langle -1, 1, 1 \rangle \cdot \langle 2, -1, 0 \rangle}{|\langle 2, -1, 0 \rangle|} \\ &= \frac{-3}{\sqrt{5}} \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech