

# Multivariable and Vector Calculus

Alvin Lin

August 2017 - December 2017

## Vectors

$$A(a_1, a_2, a_3)$$

$$B(b_1, b_2, b_3)$$

$$\vec{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

## Vector Operations

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\lambda \vec{a} = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

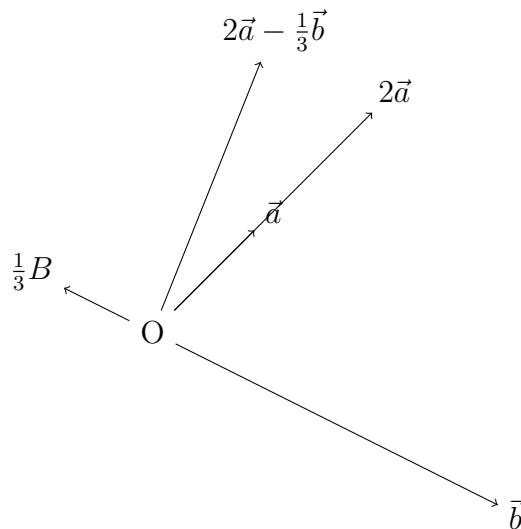
$$|\lambda \vec{a}| = |\lambda| |\vec{a}|$$

$$\lambda(\vec{a} + \vec{b}) = \lambda \vec{a} + \lambda \vec{b}$$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = |\vec{a}| \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|} \right\rangle = |\vec{a}| \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

## Example

Draw  $2\vec{a} - \frac{1}{3}\vec{b}$ :



**Example**

Given the vector of length 7 which makes an angle of  $15^\circ$  with  $\langle 3, 3 \rangle$ .

$$\vec{a} = 7\langle \cos(30), \cos(60) \rangle = 7\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \langle \frac{7\sqrt{3}}{2}, \frac{7}{2} \rangle$$

Alternate Solution:

$$\vec{a} = 7\langle \cos(60), \cos(30) \rangle = 7\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \langle \frac{7}{2}, \frac{7\sqrt{3}}{2} \rangle$$

**Example**

A pilot steers a plane at 500mph towards N 30 E, while the wind blows at 200mph from N 45 W. Find the true course and speed.

$$\begin{aligned} \vec{t} &= \overrightarrow{pilot} + \overrightarrow{wind} \\ &= 500\langle \cos(60), \cos(30) \rangle + 200\langle \cos(45), -\cos(45) \rangle \\ &= \langle 250, 250\sqrt{3} \rangle + \langle 100\sqrt{2}, -100\sqrt{2} \rangle \\ &= \langle 250 + 100\sqrt{2}, 250\sqrt{3} - 100\sqrt{2} \rangle \end{aligned}$$

### Example

Give a vector of length 3, tangent to  $y = 2 \sin(x)$  at  $\frac{\pi}{6}, 1$ .

$$\begin{aligned}y' &= 2 \cos(x) \\y'(\frac{\pi}{6}) &= \sqrt{3} \\ \vec{v} &= \langle 1, \sqrt{3} \rangle \\ |\vec{v}| &= 2 \\ \frac{3\vec{v}}{|\vec{v}|} &= \langle \frac{3}{2}, \frac{\sqrt{3}}{2} \rangle\end{aligned}$$

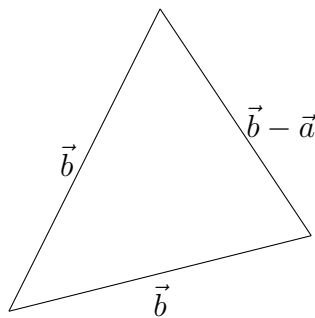
### Dot/scalar Product

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = \sum_i^n a_i b_i$$

Properties:

$$\begin{aligned}\vec{a} \cdot \vec{a} &= |\vec{a}|^2 \\ \vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{a} \\ \vec{a} \cdot (\vec{b} + \vec{c}) &= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \\ (\lambda \vec{a}) \cdot \vec{b} &= \lambda(\vec{a} \cdot \vec{b}) \\ \vec{a} \cdot \vec{b} &= 0 \quad (\vec{a} \perp \vec{b})\end{aligned}$$

Law of Cosines:



$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos(\theta)$$

Using the vector properties:

$$\begin{aligned}(\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) &= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2|\vec{a}||\vec{b}|\cos\theta \\ \vec{a} \cdot \vec{b} &= |\vec{a}||\vec{b}|\cos\theta\end{aligned}$$

### Example

$$\vec{a} = \langle 1, 2, 3 \rangle, \vec{b} = \langle 2, x, 4 \rangle$$

For what  $x$  is  $\vec{a} \perp \vec{b}$ :

$$\begin{aligned}\vec{a} \cdot \vec{b} &= \langle 1, 2, 3 \rangle \cdot \langle 2, x, 4 \rangle \\ &= 2 + 2x + 12 = 0 \\ 2x &= -14 \\ x &= -7\end{aligned}$$

### Example

Given a triangle ABC where A(1,1,2), B(2,4,3), C(3,5,8), find angle  $\angle CAB$ . Let  $\alpha = \angle CAB$ .

$$\begin{aligned}\cos\alpha &= \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} \\ &= \frac{\langle 1, 3, 1 \rangle \cdot \langle 2, 4, 6 \rangle}{|\langle 1, 3, 1 \rangle||\langle 2, 4, 6 \rangle|} \\ &= \frac{2 + 12 + 6}{\sqrt{11}\sqrt{56}} \\ &= \frac{20}{\sqrt{11}\sqrt{56}} \\ \alpha &= \cos^{-1}\left(\frac{20}{\sqrt{11}\sqrt{56}}\right)\end{aligned}$$

### Example

Given two lines:

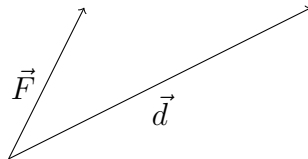
$$\begin{aligned}l_1 : x - 2y &= 3 \\ l_2 : y &= 4x + 7\end{aligned}$$

Find their angle of intersection: We can take two vectors parallel to the the lines and find the angle between them:

$$\langle 2, 1 \rangle, \langle 1, 4 \rangle$$

$$\begin{aligned} \cos \theta &= \frac{\langle 2, 1 \rangle \cdot \langle 1, 4 \rangle}{|\langle 2, 1 \rangle| |\langle 1, 4 \rangle|} \\ &= \frac{6}{\sqrt{5}\sqrt{17}} \\ \theta &= \cos^{-1}\left(\frac{6}{\sqrt{5}\sqrt{17}}\right) \end{aligned}$$

## Force and Displacement



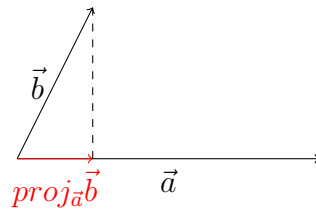
$$W = |\vec{F}| \cos \theta |\vec{d}| = \vec{F} \cdot \vec{d}$$

### Example

A sled is pulled 100ft up a hill with an angle of  $30^\circ$ . The angle between the handle and the sled is  $15^\circ$ . Given that the sled is 20 pounds, what is the amount of work done?

$$\begin{aligned} W &= \vec{F} \cdot \vec{d} \\ &= 20 \langle \cos 30, \cos 60 \rangle \cdot 100 \langle \cos 45, \cos 45 \rangle \\ &= 2000 \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle \\ &= 2000 \left( \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \right) \\ &= 500(\sqrt{6} + \sqrt{2}) \end{aligned}$$

## Vector Projections

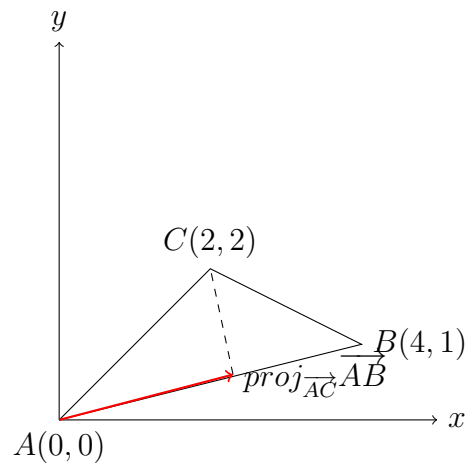


The scalar project of  $\vec{b}$  onto  $\vec{a}$  is  $comp_{\vec{a}}\vec{b}$ .

$$\begin{aligned}comp_{\vec{a}}\vec{b} &= |\vec{b}| \cos \theta \\ &= \frac{|\vec{b}| \cos \theta |\vec{a}|}{|\vec{a}|} \\ &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\end{aligned}$$

$$\begin{aligned}proj_{\vec{a}}\vec{b} &= comp_{\vec{a}}\vec{b} \frac{\vec{a}}{|\vec{a}|} \\ &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|} \\ &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}\end{aligned}$$

### Example



$$\begin{aligned}
 \text{comp}_{\vec{AC}} \vec{AB} &= \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AC}|} \\
 &= \frac{\langle 2, 2 \rangle \cdot \langle 4, 1 \rangle}{\sqrt{17}} \\
 &= \frac{10}{\sqrt{17}}
 \end{aligned}$$

## Cross Product

$$\begin{aligned}
 \vec{a} \times \vec{b} &= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \\
 &= \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \\
 &= i \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - j \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + k \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}
 \end{aligned}$$

Properties:

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

The magnitude of the cross product of  $\vec{a}$  and  $\vec{b}$  is the area of a parallelogram with sides  $\vec{a}$  and  $\vec{b}$ . Since we know this, the area of a triangle ABC is  $\frac{1}{2} |\vec{AB} \times \vec{AC}|$ .

$$\vec{a} \times \vec{b} = \vec{0} \iff \vec{a} \parallel \vec{b}$$

If  $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ , then  $\vec{a} \times \vec{b}$  is perpendicular to the plane described by  $\vec{a}$  and  $\vec{b}$ .

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = 0 \rightarrow \vec{a} \times \vec{b} \perp \vec{a}$$

Vice versa:

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0 \rightarrow \vec{a} \times \vec{b} \perp \vec{b}$$

Additionally:

$$\begin{aligned}
 \vec{a} \times \vec{b} &= -(\vec{b} \times \vec{a}) \\
 \vec{a} \times (\vec{b} + \vec{c}) &= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \\
 (\lambda \vec{a}) \times \vec{b} &= \lambda(\vec{a} \times \vec{b})
 \end{aligned}$$

### Example

Given a parallelepiped described by the vectors  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ :

$$\begin{aligned} V &= \text{area of base times height} \\ &= |\vec{a} \times \vec{b}|h \\ &= |\vec{a} \times \vec{b}|c \cos \angle(\vec{c}, \vec{a} \times \vec{b}) \\ &= (\vec{a} \times \vec{b}) \cdot \vec{c} \\ &= \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \end{aligned}$$

Consider the following four points: A(1,1,1), B(2,1,3), C(1,2,4), D(2,2,1). Are they coplanar? We can check if  $\vec{AB} \times \vec{AC} \parallel \vec{DC} \times \vec{DB}$ , or we can check if the volume of the parallelepiped described by  $\vec{AB}$ ,  $\vec{AC}$ ,  $\vec{AD}$  is 0.

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} = -1(-3) + 0(-3) + 2(-1) = -5 \neq 0$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)