# Counting

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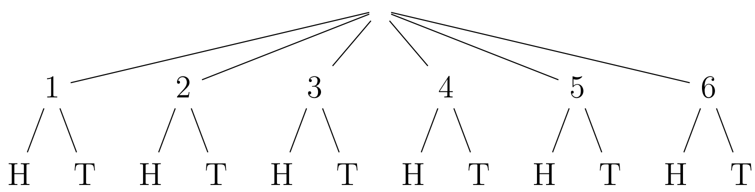
## Counting

Counting is a higher level math topic involving subtle counting rules, applying logic, and set theory. It involves questions such as:

1. How many license plates have the form AAA-DDDD where A is an uppercase alphabetical character and D is a digit?
2. Jessica, Tim, Dan, and Mary are lining up to purchase tickets at a movie theater. In how many ways can they line up?
3. Two people from a group of 10 are being selected to co-chair an upcoming event. How many ways are there to select 2 people from 10?
4. In a binary string of length 5, how many strings contain exactly two 1's somewhere in the string?
5. 75 students are enrolled in math or english. If 40 students are enrolled in math and 50 students are enrolled in english then how many students are enrolled in both?

### Tree Diagrams

A **tree diagram** is a useful visual tool for enumerating outcomes. For example, suppose we wish to list all of the outcomes from rolling a die and then flipping a coin. The tree diagram begins at a point and then splits into 6 directions. Then each branches off into 2 directions from flipping a coin.



## Product Rule

In a string of length $k$, if there are $n_1$ ways to fill position 1, $n_2$ ways to fill position 2, … and $n_k$ ways to fill position $k$ then there are $n_1 \cdot n_2 \cdot \dots \cdot n_k$ strings altogether.

## Addition Rule

If a set $S$ can be divided up into $k$ non-overlapping subsets $T_1, T_2, \dots, T_k$ then $|S| = |T_1| + |T_2| + \dots + |T_k|$.

## Opposite Rule

Suppose $S$ consists of a set of objects chosen from a larger universe of objects $\mathbb{U}$. Then $|S| = |\mathbb{U}| - |\overline{S}|$.

## General Pigeonhole Principle

Given $N$ objects and $k$ boxes or categories, there is at least one box with at least $M$ objects where $M = \lceil \frac{N}{k} \rceil$.

## Permutations and Combinations

- The **factorial** of a number $n$, or $n$ factorial, is the product of all the numbers from $n$ down to 1. Additionally, 0 factorial is 1 by definition and $n$ factorial is denoted by $n!$. For example, $3! = 3 \cdot 2 \cdot 1 = 6$.

- An **r permutation of** $n$ **distinct objects** is an ordered arrangement of $r$ distinct objects from the original $n$ objects. For example, the 2 permutations of 'abc' are 'ab', 'ba', 'ac', 'ca', 'bc', and 'cb'. The total number of $r$ permutations from $n$ objects is denoted by $P(n, r)$ or ${}_nP_r$. For example, $P(3, 2) = 6$ since we found 6 strings of length 2 when selected from the 3 distinct letters in 'abc'.

- An **r combination of $n$ distinct objects** is an unordered selection of $r$ distinct objects from the original $n$ objects. For example, the 2 combinations of 'abc' are $\{a, b\}, \{a, c\}, \ and \ \{b, c\}$. The total number of $r$ combinations from $n$ objects is denoted by $C(n, r)$ or ${}_nC_r$. For example, $C(3, 2) = 3$ since we found 3 sets of size 2 when selecting from the 3 distinct letters in 'abc'.

- **$k$ to 1 Correspondences**: If there are $k$ objects in set A that correspond to 1 object in set B, then $|A| = k \cdot |B|$.

- **Permutation with Repetition**: In a string of length $n$ with $r_1$ letters of type 1, $r_2$ letters of type 2, ..., and $r_k$ letters of type $k$, then there are

$$\frac{n!}{(r_1!)(r_2!)\cdots(r_k!)}$$

  distinguishable permutations when rearranging all of the letters in the original string.

- **Combination with Repetition**: In making an unordered selection from $n$ objects with $n_1$ objects of type 1, $n_2$ objects of type 2, ..., and $n_k$ objects of type $k$, then there are

$${}_{n_1}C_{r_1} \cdot_{n_2} C_{r_2} \cdots_{n_k} C_{r_k}$$

  distinguishable combinations when selecting $r_1$ objects of type 1, $r_2$ objects of type 2, ..., and $r_k$ objects of type $k$.

## Facts about $P(n, r)$ and $C(n, r)$

1. $P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$
2. $P(n, r) = \frac{n!}{(n-r)!}$
3. $P(n, r) = r!_nC_r$
4. $C(n, r) = \frac{n(n-1)\ldots(n-r+1)}{r!}$
5. $C(n, r) = \frac{n!}{(n-r)!r!}$
6. $P(n, r) = {}_nP_r$
7. $C(n, r) = {}_nC_r = \binom{n}{r}$
8. $C(n, 0) = 1 = \frac{n!1}{n!0!} = \frac{1}{1}$

9. $C(n, n) = 1$
10. $C(n, r) = C(n, n - r)$

## Binomial Theorem

$$
\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= 1x + 1y \\
(x+y)^2 &= 1x^2 + 2xy + 1y^2 \\
(x+y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
(x+y)^n &= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n \\
(x+y)^n &= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i
\end{aligned}
$$

## Distributing Objects

There are $k^n$ ways to distribute $n$ distinguishable objects amongst $k$ boxes. However, there are only $\binom{n+k-1}{k-1}$ ways to distribute $n$ indistinguishable objects amongst $k$ boxes.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech