Sets

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Mathematical Induction

Mathematical inductions prove statements that depend on a positive integer n. In general, we use it to prove $\forall n P(n)$ where we normally have $\mathbb{U} = \mathbb{Z} \geq a$ for $a = \{0, 1, 2, 3, 4, 5, \ldots\}$. Unless otherwise specified, we assume $a = 1$.

Examples

For all $n > 0$:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

For all $n > 0$:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

For all $n > 0$, 2 divides $n^2 + n$:

$$n^2 + n \equiv 0 \pmod 2$$

For all $n > 0$:

$$4^{n+1} + 5^{2n-1} \equiv 0 \pmod{21}$$

Process

1. Identify the exact $P(n)$ statement.

2. Write out $P(k)$ in the induction hypothesis.

3. Write out $P(k+1)$ in the induction.
4. Think about how $P(k+1)$ relates to $P(k)$.
5. Once you apply the induction hypothesis in the induction, the rest is algebra.

With the math induction argument, we have:

$$\begin{array}{rl} & P(1) \\ & \underline{\forall k P(k) \rightarrow P(k+1)} \\ \therefore & \forall n P(n) \end{array}$$

Using Modus Ponens:

$$\begin{array}{rl} & P(1) \\ & \underline{P(1) \rightarrow P(2)} \\ \therefore & P(2) \end{array}$$

By hypothetical syllogism:

$$\begin{array}{rl} & P(1) \rightarrow P(2) \\ & \underline{P(2) \rightarrow P(3)} \\ \therefore & P(1) \rightarrow P(3) \end{array}$$

Example

Prove $\forall n \sum_{i=1}^{n}(2i-1) = n^2$.

1. Basis: $P(n) = \sum_{i=1}^{n}(2i-1) = n^2$
2. Show $P(1)$ is true:

$$\begin{aligned} \sum_{i=1}^{1}(2i-1) &= 1^1 \\ (2 \cdot 1 - 1) &= 1^2 \\ 2 - 1 &= 1 \end{aligned}$$

3. Induction hypothesis, assume $P(k)$ is true for some $k \geq 1$. Assume $\sum_{i=1}^{k}(2i - 1) = k^2$.

4. Induction (we need to prove this):

$$P(k+1) = \sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$
$$\sum_{i=1}^{k+1}(2i-1) = k^2 + 2k + 1$$

5. Proof:

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + \sum_{k}^{k+1}(2i-1) \\
&= \sum_{i=1}^{k}(2i-1) + (2(k+1)-1) \\
&= k^2 + (2k+2-1) \\
&= k^2 + 2k - 1
\end{aligned}$$

Example

Prove $\forall n, n^3 - n \equiv 0 (\text{mod } 3)$. Let $P(n)$ be the idea $n^3 - n \equiv 0 (\text{mod } 3)$.

1. Basis: $P(0)$ is the statement $0^3 - 0 \equiv 0 (\text{mod } 3)$. We get $0 \equiv 0 (\text{mod } 3)$ from simplifying this statement, which is true.

2. Induction hypothesis: assume that $P(k)$ is true for some $k \geq 0$. This means that $k^3 - k \equiv 0 (\text{mod } 3)$ is true.

3. Induction (we need to prove this):

$$(k+1)^3 - (k+1) \equiv 0 \pmod 3$$

4. Proof:

$$\begin{aligned}
(k+1)^3 - (k+1) &\equiv k^3 + 3k^2 + 3k + 1 - k - 1 \pmod 3 \\
&\equiv (k^3 - k) + 3k^2 + 3k \\
&\equiv 0 + 0k^2 + 0k \equiv 0
\end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech