Vectors and Matrices

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Vectors and Matrices

An $m \times n$ **matrix** is a two-dimensional array of numbers consisting of m rows and n columns. For example:

$$A_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

This is a 2×3 matrix with $a_{1,1} = 1$ and $a_{2,1} = 4$. In general, $a_{i,j}$ is the row i, column j entry of matrix A. An $m \times n$ matrix has dimension $m \times n$. If $n = m$, then we say that A is a **square matrix**.

Matrix Operations

Suppose A and B are both $m \times n$ matrices and c is a real number.

- Matrix Addition: $A + B = C$ where $c_{i,j} = a_{i,j} + b_{i,j}$
- Matrix Subtraction: $A - B = D$ where $d_{i,j} = a_{i,j} - b_{i,j}$
- Scalar Multiplication: $c \cdot A = F$ where $f_{i,j} = c \cdot a_{i,j}$
- Matrix Multiplication: $A \cdot B = M$ where $m_{i,j} = A_{row\ i} \cdot B_{col\ j}$. Suppose A is a $m \times n$ matrix and B is a $n \times r$ matrix and $A \cdot B = M$ where M is a $m \times r$ matrix:

  $$m_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

  Matrix multiplication is not commutative, usually $A \cdot B \neq B \cdot A$.

- Inverse of a 2×2 matrix: Given $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$, find $A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, such that $A \cdot A^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ ax + bz &= 1 \\ ay + bw &= 0 \\ cx + dz &= 0 \\ cy + dw &= 1 \\ A^{-1} &= \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

- Determinant of a 2×2 matrix: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = ad - bc$

Identity of a 2×2 Matrix

The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ behaves like 1 in terms of multiplication.

Examples

$$\begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -7 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ -5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 15 & -3 \end{bmatrix} - \begin{bmatrix} 3 & 10 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -10 \\ 11 & -7 \end{bmatrix}$$

$$4 \cdot \begin{bmatrix} 5 & 1 \\ 3 & 2 \\ -1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 4 \\ 12 & 8 \\ -4 & 40 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 13 \\ 17 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solving a System of Linear Equations

$$
\begin{aligned}
1x + 2y &= 5 \\
3x + 4y &= 8 \\
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 5 \\ 8 \end{bmatrix} \\
A \cdot \vec{v} &= \vec{w} \\
A^{-1} A \vec{v} &= A^{-1} \vec{w} \\
I \vec{v} &= A^{-1} \vec{w} \\
\vec{v} &= A^{-1} \vec{w}
\end{aligned}
$$

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \vec{v} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} -10 + 18 \\ \frac{15}{2} - \frac{8}{2} \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{7}{2} \end{bmatrix}
$$

$$
x = -2 \quad y = \frac{7}{2}
$$

Vectors

A **vector** is a one-dimensional array of numbers which may be thought of as a $1 \times n$ or $n \times 1$ matrix. For example:

$$
\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
$$

This is a vector with $v_1 = 1$, $v_2 = 2$, and $v_3 = 3$. A vector is a 1-dimensional array of numbers which encodes magnitude and direction. A vector with length 3 with a heading of $45°$ in 2D is equivalent to:

$$
\vec{v} = \begin{bmatrix} \frac{3\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \end{bmatrix}
$$

Vector Operations

- Vector Length: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
- Dot Product: $\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$

- Angle between Vectors: $\vec{v} \bullet \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos(\theta)$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech