Sets

Alvin Lin

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Countable and Uncountable Sets

Recall for a set S, $|S|$ is the cardinality of S, If S has a finite number of objects, then $|S| \in N$. Consider the following six sets:

- E is the set of positive even numbers less than or equal to 100.
- $\mathbb{Z}$ is the set of all integers.
- U is the set of all real numbers between 0 and 1.
- P is the set of all prime numbers less than or equal to 50.
- $\mathbb{Q}$ is the set of all rational numbers.
- $\mathbb{R}$ is the set of all real numbers.

Both E and P are finite sets with $|E| = 50$ and $|P| = 15$. The remaining four sets are all infinite. If $f : A \to B$ is 1-1, then $|A| \leq |B|$. Recall from functions that f is 1-1 if every output is uniquely associated to 1 input. Also, for real-valued functions, f is 1-1 if it passed the horizontal line test. If $A \subseteq B$, then $|A| \leq |B|$ by using $f(x) = x$ where $f : A \to B$.

Schröder-Bernstein Theorem

If $f : A \to B$ is 1-1 and $g : B \to A$ is 1-1, then $|A| = |B|$. Using the above example, $|U| = |\mathbb{R}|$ since $f(x) = \frac{\tan^{-1}(x)+\frac{\pi}{2}}{\pi}$ where $f : \mathbb{R} \to U$ is 1-1 and $g(x) = x$ where $g : U \to \mathbb{R}$ is 1-1.

Countably Infinite Sets

Set A is **countably infinite** if $|A| = |\mathbb{Z}^+|$. Equivalently, there are 1-1 functions f, g such that $f : A \to \mathbb{Z}^+$ and $g : \mathbb{Z}^+ \to A$. There is a good enumeration of $A = \{a_1, a_2, \ldots$ such that every number in A shows up once and every number appears with a finite index. A set is **countable** if it is a finite set or countably infinite.
Countably Infinite Sets: $\mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}$
Uncountable Sets: $\mathbb{R}, (0, 1), P(\mathbb{Z}) : subsets\ of\ \mathbb{Z}$
If A, B are both countable, then $A \cup B$ is countable.

Enumerating a Countable Set

A good enumeration for a set A involves listing out the values in A with a predictable pattern so that every value in A is listed at a finite step. For example, $\mathbb{Z} = \{0, 1, 2, 3, \ldots, -1, -2, -3, \ldots\}$ is not a good enumeration as we would have to go through all infinitely many positive integers before seeing any negative integers. A good enumeration for $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ so that both positive integers and negative integers are reached in a finite number of steps.

Subsets of Sets

Suppose $A \subseteq B$:

- If B is countable, then A is countable as well.
- If B is uncountable, then A is uncountable as well.

Uncountable Sets

Two sets A and B have the same **cardinality**, written $A \sim B$, if there is a bijection between them. Let A be a set:

- A is **countable** if it is finite: $|A| \in \mathbb{Z}$
- A is **countably infinite** if $A \sim \mathbb{Z}^+$
- A is **uncountable** if it is not countable.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech