Sets

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Countable and Uncountable Sets

Recall for a set S, |S| is the cardinality of S, If S has a finite number of objects, then $|S| \in N$. Consider the following six sets:

- E is the set of positive even numbers less than or equal to 100.
- \mathbb{Z} is the set of all integers.
- \bullet *U* is the set of all real numbers between 0 and 1.
- P is the set of all prime numbers less than or equal to 50.
- Q is the set of all rational numbers.
- \bullet \mathbb{R} is the set of all real numbers.

Both E and P are finite sets with |E| = 50 and |P| = 15. The remaining four sets are all infinite. If $f: A \to B$ is 1-1, then $|A| \le |B|$. Recall from functions that f is 1-1 if every output is uniquely associated to 1 input. Also, for real-valued functions, f is 1-1 if it passed the horizontal line test. If $A \subseteq B$, then $|A| \le |B|$ by using f(x) = x where $f: A \to B$.

Schröder-Bernstein Theorem

If $f:A\to B$ is 1-1 and $g:B\to A$ is 1-1, then |A|=|B|. Using the above example, $|U|=|\mathbb{R}|$ since $f(x)=\frac{\tan^{-1}(x)+\frac{\pi}{2}}{\pi}$ where $f:\mathbb{R}\to U$ is 1-1 and g(x)=x where $g:U\to\mathbb{R}$ is 1-1.

Countably Infinite Sets

Set A is **countably infinite** if $|A| = |\mathbb{Z}^+|$. Equivalently, there are 1-1 functions f, g such that $f: A \to \mathbb{Z}^+$ and $g: \mathbb{Z}^+ \to A$. There is a good enumeration of $A = \{a_1, a_2, \ldots$ such that every number in A shows up once and every number appears with a finite index. A set is **countable** if it is a finite set or countably infinite.

Countably Infinite Sets: $\mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}$ Uncountable Sets: $\mathbb{R}, (0,1), P(\mathbb{Z})$: subsets of \mathbb{Z} If A, B are both countable, then $A \cup B$ is countable.

Enumerating a Countable Set

A good enumeration for a set A involves listing out the values in A with a predictable pattern so that every value in A is listed at a finite step. For example, $\mathbb{Z} = \{0, 1, 2, 3, \ldots, -1, -2, -3, \ldots\}$ is not a good enumeration as we would have to go through all infinitely many positive integers before seeing any negative integers. A good enumeration for $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ so that both positive integers and negative integers are reached in a finite number of steps.

Subsets of Sets

Suppose $A \subseteq B$:

- If B is countable, then A is countable as well.
- If B is uncountable, then A is uncountable as well.

Uncountable Sets

Two sets A and B have the same **cardinality**, written $A \sim B$, if there is a bijection between them. Let A be a set:

- A is **countable** if it is finite: $|A| \in \mathbb{Z}$
- A is countably infinite if $A \sim \mathbb{Z}^+$
- A is uncountable if it is not countable.

You can find all my notes at http://omgimanerd.tech/notes. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech