

# Sets

Alvin Lin

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## Functions

Let  $A$  and  $B$  be sets. A **function**  $f$  is a rule that assigns to each element  $x \in A$  exactly one element  $y \in B$ , written  $y = f(x)$ .  $A$  is called the **domain** while  $B$  is called the **codomain**. The **range** of  $f$ , denoted  $\text{ran}(f)$ :

$$\text{ran}(f) = \{f(x) \in B \mid x \in A\}$$

$$\text{ran}(f) \subseteq B$$

### Example

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = x^2$ .

- Domain:  $\mathbb{R}$
- Codomain:  $\mathbb{R}$
- Range:  $[0, \infty)$

### Example

Let  $g : \mathbb{R} \rightarrow [0, \infty)$  be defined as  $g(x) = x^2$ .

- Domain:  $\mathbb{R}$
- Codomain:  $[0, \infty)$
- Range:  $[0, \infty)$

## Function Equality

Two functions,  $f : A \rightarrow B$  and  $g : C \rightarrow D$  are **equal**, denoted  $f = g$ , if:

- $A = C$
- $B = D$
- $f(x) = g(x) \quad \forall x \in A$

## Absolute Value

The **absolute value** of  $x \in \mathbb{R}$ , written  $abs(x)$  or  $|x|$ , is a piecewise function defined as:

$$abs(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$
$$abs : \mathbb{R} \rightarrow \mathbb{R}$$

- Domain:  $\mathbb{R}$
- Codomain:  $\mathbb{R}$
- Range:  $[0, \infty)$

## Example

Let  $X$  be a set,  $P(X)$  be the power set of  $X$ , and  $S \subseteq X$ . Let  $f : P(X) \rightarrow P(X)$  be defined as  $f(A) = A \cup S$ .

- Domain:  $P(X)$
- Codomain:  $P(X)$
- Range:  $\{B \subseteq X \mid B \supseteq S\}$

## Function Images and Preimages

Let  $f : A \rightarrow B$  be a function. If  $S \subseteq A$ , then the **image** of  $S$ , denoted  $f(S)$ , is defined as:

$$f(S) = \{f(x) : x \in S\}$$

If  $T \subseteq B$ , then the **preimage** of  $S$ , denoted  $f^{-1}(S)$ , is defined as:

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\}$$

## Example

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2$ . Determine:

1.  $f([-1, 2]) = [0, 4]$
2.  $f^{-1}(\{2\}) = \{\sqrt{2}, -\sqrt{2}\}$
3.  $f^{-1}([-1, 2])$

$$\begin{aligned}f^{-1}(\{-1\}) &= \emptyset \\f^{-1}([-1, 0]) &= \emptyset \\f^{-1}(\{a, b\}) &= f^{-1}(\{a\}) \cup f^{-1}(\{b\}) \\f^{-1}([-1, 2]) &= f^{-1}([-1, 0]) \cup f^{-1}([0, 2]) \\&= \emptyset \cup [-\sqrt{2}, \sqrt{2}] \\&= [-\sqrt{2}, \sqrt{2}]\end{aligned}$$

## One-To-One and Onto

Let  $f : A \rightarrow B$ ,  $f$  is **one-to-one** if distinct elements in  $A$  have distinct images in  $B$ .  
To show a function is one-to-one:

$$\forall x, y \in A (x \neq y \rightarrow f(x) \neq f(y))$$

Contrapositive:

$$\forall x, y \in A (f(x) = f(y) \rightarrow x = y)$$

## Onto

Let  $f : A \rightarrow B$ ,  $f$  is **onto** if the range and codomain are the same.

$$B = \text{ran}(f) = f(A)$$

## Example

Let  $x \in \mathbb{R}$ . The greatest integer of  $x$ , denoted  $\lfloor x \rfloor$ , is the largest integer less than or equal to  $x$ .  $\lfloor x \rfloor = \text{floor}(x) = \text{floor} : \mathbb{R} \rightarrow \mathbb{R}$ .

$$\begin{aligned}\lfloor 2.1 \rfloor &= 2 \\ \lfloor \pi \rfloor &= 3 \\ \lfloor -2.1 \rfloor &= -3\end{aligned}$$

Is  $\lfloor x \rfloor$  one-to-one?

$$\begin{aligned}\lfloor 1.1 \rfloor &= \lfloor 1 \rfloor \\ 1.1 &\neq 1\end{aligned}$$

Therefore, it is not one-to-one.

Is  $\lfloor x \rfloor$  onto?

No, there is no  $x \in \mathbb{R}$  such that  $\lfloor x \rfloor = 2.5$ .

### Example

Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(n) = n^3$ . Is  $f$  one-to-one?

- Suppose  $x, y \in \mathbb{Z}$  and  $f(x) = f(y)$ .
- This implies  $x^3 = y^3$ .
- Taking the cube root yields  $x = y$ .

Is  $f$  onto?

No, since  $2 \in \mathbb{Z}$  but there is no  $x \in \mathbb{Z}$  such that  $x^3 = 2$ .

### Example

Let  $X$  be a set and  $X \subseteq X$ . Define  $f : P(X) \rightarrow P(X)$  as  $f(A) = A \cup X$ . Is  $f$  one-to-one?

No, since  $f(\emptyset) = f(X)$  but  $\emptyset \neq X$ .

$$\begin{aligned}f(\emptyset) &= \emptyset \cup X = X \\ f(X) &= X \cup X = X\end{aligned}$$

### Example

Let  $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(m, n) = m + n$ .

Is  $f$  one-to-one?

No,  $f(2, 3) = f(3, 2)$  but  $(2, 3) \neq (3, 2)$

Is  $f$  onto?

Yes, take any  $n \in \mathbb{Z}$ . Find an ordered pair in  $\mathbb{Z} \times \mathbb{Z}$  that maps to it. Since  $f(n - 1, 1) = (n - 1) + 1 = n$  and  $(n - 1, 1) \in \mathbb{Z} \times \mathbb{Z}$ , then  $f$  is onto.

## Identity Mapping

Let  $A$  be a set. The function  $1_A : A \rightarrow A$  is defined as:

$$1_A(x) = x$$

This is called the **identity mapping**.

## Bijections

A function is a **bijection** if it is one-to-one and onto. A “bijection” is a “one-to-one correspondence”.

## Composition

Let  $f : A \rightarrow A$  and  $g : B \rightarrow C$ . The **composition** of  $f$  and  $g$ , denoted  $g \circ f$ , is  $g \circ f : A \rightarrow C$  and defined as:

$$(g \circ f)(x) = g(f(x))$$

## Inverse

Let  $f$  be a bijection. ( $f : A \rightarrow B$ ). The **inverse** of  $f$  is the unique function  $g : B \rightarrow A$  such that  $g \circ f = 1_A$  and  $f \circ g = 1_B$ . Traditionally, we represent  $g$  as  $f^{-1}$ . Note that  $f^{-1} \neq \frac{1}{f}$ .

## Example

Let  $A = \{1, 2, 3\}$ ,  $B = \{x, y, z\}$ ,  $C = \{\alpha, \beta, \gamma\}$ . Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be defined as:

$$\begin{aligned} f &= \{(1, x), (2, y), (3, x)\} \\ g &= \{(x, \gamma), (y, \alpha), (z, \beta)\} \end{aligned}$$

Find  $f \circ g$ :

$$\begin{aligned} f \circ g &= \text{undefined} \\ g \circ f &= \{(1, \gamma), (2, \alpha), (3, \beta)\} \end{aligned}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)