Propositional Logic

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Propositional Logic

Logic gives precise meaning to mathematical statements. A **proposition** is a statement that declares a fact. Ex:

- $1 + 1 = 2$
- $1 + 1 = 3$
- The sky is blue
- Today is Monday

Note that the first and third are true, while the second and fourth proposition are false. Propositions can be either true or false.

We use letters to represent propositions, generally p, q, r, s, t. Truth values of a proposition are denoted as True/False, T/F, or 1/0.

The following are not propositions:

- What time is it?
- $x + 1 = 9$
- $x^2 + y^2 = z^2$

Negation

Let p be a proposition. The **negation** of p, denoted $\neg p$, is the statement "it is not the case p", read as "not p".

p	$\neg p$
T	F
F	T

Conjunction

Let p and q be propositions. The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition p *and* q. This conjunction is only true when both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

Let p and q be propositions. The **disjunction** of p and q, denoted by $p \vee q$, is the proposition p *or* q. This conjunction is true when either p or q are true, or both.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

Let p and q be propositions. The **exclusive or**, denoted $p \oplus q$, is the proposition with the following truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements

Let p and q be propositions. The conditional statement $p \rightarrow q$ is the proposition "if p then q". This is false when p is true and q is false, and true otherwise.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In English, the following mean that $p \to q$:

- if p, then q
- p implies q
- p is sufficient for q
- p whenever q
- q is necessary for p

Converse, Contrapositive, and Inverse

Given $p \to q$, we could consider the following:

- The **converse** of $p \to q$ is $q \to p$.
- The **contrapositive** of $p \to q$ is $\neg q \to \neg p$.
- The **inverse** of $p \to q$ is $\neg p \to \neg q$.

p	q	$p \to q$	$q \to p$	$\neg q \to \neg p$	$\neg p \to \neg q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Propositions that have the same truth table are called **equivalent**. $p \to q$ and $\neg q \to \neg p$ are always equivalent. This is useful in mathematical proofs since it may be easier to prove the contrapositive of a proposition.

Biconditionals

Let p and q be propositions. The **biconditional** statement $p \leftrightarrow q$ is the proposition "p if and only if q". $p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Precedence of Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee \oplus$	3
$\to$	4
$\leftrightarrow$	5

$$\neg p \wedge q := (\neg p) \wedge q$$

Example

Find the truth table for $(p \vee \neg q) \to (p \wedge q)$.

p	q	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	T	F	F

Example 2

Suppose we have an agreement:
"If there is a horse in the library, then I will pay you \$10"
Let $p =$ there is a horse in the library.
Let $q =$ you get \$10.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech