

Section 7.4

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Calculus II: August 2016 - December 2016

Exercise 7

$$\int \frac{x^4}{x-1} dx$$
$$\begin{array}{r} x^3 + x^2 + x + 1 \\ x-1 \overline{) x^4} \\ \underline{-x^4 + x^3} \\ x^3 \\ \underline{-x^3 + x^2} \\ x^2 \\ \underline{-x^2 + x} \\ x \\ \underline{-x + 1} \\ 1 \end{array}$$

$$\int \frac{x^4}{x-1} dx = \int x^3 + x^2 + x + 1 + \frac{1}{x-1} dx$$
$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x+1| + C$$

Exercise 11

$$\int_0^1 \frac{2}{2x^2 + 3x + 1} dx$$
$$\frac{2}{2x^2 + 3x + 1} = \frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$
$$(2x+1)(x+1) \frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1} (2x+1)(x+1)$$
$$2 = A(x+1) + B(2x+1)$$
$$2 = Ax + A + 2Bx + B = (A+2B)x + (A+B)$$

$$\begin{aligned}
A + 2B &= 0 & A + B &= 2 \\
A + 2(2 - A) &= A + 4 - 2A = 0 \\
A &= 4 & B &= -2 \\
\frac{2}{2x^2 + 3x + 1} &= \frac{4}{2x + 1} - \frac{2}{x + 1}
\end{aligned}$$

Solve it as an indefinite integral:

$$\begin{aligned}
\int \frac{2}{2x^2 + 3x + 1} dx &= \int \frac{4}{2x + 1} - \frac{2}{x + 1} dx \\
2 \int \frac{2}{2x + 1} dx - 2 \int \frac{1}{x + 1} dx & \\
2 \int \frac{2}{2x + 1} dx - 2 \int \frac{1}{x + 1} dx & \\
2 \ln |2x + 1| - 2 \ln |x + 1| + C &
\end{aligned}$$

Now we apply the original limits of integration:

$$\begin{aligned}
&\left[2 \ln |2x + 1| - 2 \ln |x + 1| \right]_0^1 \\
&2 \ln |2(1) + 1| - 2 \ln |1 + 1| - (2 \ln |2(0) + 1| - 2 \ln |0 + 1|) \\
&2 \ln |3| - 2 \ln |2| - (0) \\
&= \ln \left| \frac{9}{4} \right|
\end{aligned}$$

Exercise 19

$$\begin{aligned}
&\int_0^1 \frac{x^2 + x + 1}{(x + 1)^2(x + 2)} dx \\
&\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{Ax + B}{(x + 1)^2} + \frac{D}{x + 2} \\
(x + 1)^2(x + 2) \frac{x^2 + x + 1}{(x + 1)^2(x + 2)} &= \frac{Ax + B}{(x + 1)^2} + \frac{D}{x + 2} (x + 1)^2(x + 2) \\
x^2 + x + 1 &= (Ax + B)(x + 2) + (D)(x^2 + 2x + 1) \\
x^2 + x + 1 &= Ax^2 + 2Ax + Bx + 2B + Dx^2 + 2Dx + D \\
x^2 + x + 1 &= (A + D)x^2 + (2A + B + 2D)x + (2B + D) \\
A + D &= 1 & 2A + B + 2D &= 1 & 2B + D &= 1 \\
2(1 - D) + \frac{1 - D}{2} + 2D &= 1 \\
4 - 4D + 1 - D + 4D &= 2
\end{aligned}$$

$$A = -2 \quad B = -1 \quad D = 3$$

$$\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{-2x - 1}{(x + 1)^2} + \frac{3}{x + 2}$$

Solve it as an indefinite integral:

$$\int \frac{-2x - 1}{(x + 1)^2} + \frac{3}{x + 2} dx$$

$$- \int \frac{2x + 1}{(x + 1)^2} dx + \int \frac{3}{x + 2} dx$$

$$3 \ln |x + 2| - \int \frac{2x + 1}{(x + 1)^2} dx$$

$$\text{Let : } u = x + 1 \quad x = u - 1 \quad du = dx$$

$$3 \ln |x + 2| - \int \frac{2(u - 1) + 1}{u^2} du$$

$$3 \ln |x + 2| - \int \frac{2u - 1}{u^2} du$$

$$3 \ln |x + 2| - \int \frac{2}{u} du - \int u^{-2} du$$

$$3 \ln |x + 2| - 2 \ln |u| + \frac{1}{u} + C$$

$$3 \ln |x + 2| - 2 \ln |x + 1| + \frac{1}{x + 1} + C$$

Now we apply the original limits of integration:

$$\left[3 \ln |x + 2| - 2 \ln |x + 1| + \frac{1}{x + 1} \right]_0^1$$

$$3 \ln |1 + 2| - 2 \ln |1 + 1| + \frac{1}{1 + 1} - (3 \ln |0 + 2| - 2 \ln |0 + 1| + \frac{1}{0 + 1})$$

$$3 \ln |3| - 2 \ln |2| + \frac{1}{2} - (3 \ln |2| - 0 + 1)$$

$$3 \ln |3| - 2 \ln |2| - 3 \ln |2| - \frac{1}{2}$$

Exercise 23

$$\int \frac{10}{(x-1)(x^2+9)}$$
$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+D}{x^2+9}$$
$$(x-1)(x^2+9) \frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+D}{x^2+9} (x-1)(x^2+9)$$
$$10 = A(x^2+9) + (Bx+D)(x-1)$$
$$10 = Ax^2 + 9A + Bx^2 - Bx + Dx - D$$
$$10 = (A+B)x^2 + (D-B)x + (9A-D)$$
$$A+B=0 \quad D-B=0 \quad 9A-D=10$$
$$D=9A-10 \quad 9A-10-B=0 \quad 9A-10-(-A)=10$$
$$10A=20$$
$$A=2 \quad B=-2 \quad D=8$$
$$\frac{10}{(x-1)(x^2+9)} = \frac{2}{x-1} + \frac{-2x+8}{x^2+9}$$
$$\int \frac{10}{(x-1)(x^2+9)} dx = \int \frac{2}{x-1} dx - \int \frac{2x-8}{x^2+9} dx$$
$$2 \int \frac{1}{x-1} dx - \int \frac{2x}{x^2+9} dx + \int \frac{8}{x^2+9} dx$$
$$2 \ln|x-1| - \ln|x^2+9| + \frac{8}{3} \int \frac{3}{x^2+9} dx$$
$$2 \ln|x-1| - \ln|x^2+9| + \frac{8}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

Exercise 39

$$\int \frac{\sqrt{x+1}}{x} dx$$

Let : $u = \sqrt{x+1}$

$$x = u^2 - 1$$
$$du = \frac{1}{2\sqrt{x+1}} dx$$
$$2 du \sqrt{x+1} = dx$$
$$2u du = dx$$
$$\int \frac{u}{u^2-1} 2u du$$

$$\begin{aligned}
& 2 \int \frac{u^2}{u^2 - 1} \, du \\
& 2 \int \frac{u^2 - 1 + 1}{u^2 - 1} \, du \\
& 2 \int \frac{u^2 - 1}{u^2 - 1} + \frac{1}{u^2 - 1} \, du \\
& 2 \int 1 \, du + 2 \int \frac{1}{(u - 1)(u + 1)} \, du \\
& 2u + C + 2 \int \frac{1}{(u - 1)(u + 1)} \, du \\
& \frac{1}{(u - 1)(u + 1)} = \frac{A}{u - 1} + \frac{B}{u + 1} \\
& (u - 1)(u + 1) \frac{1}{(u - 1)(u + 1)} = \frac{A}{u - 1} + \frac{B}{u + 1} (u - 1)(u + 1) \\
& 1 = A(u + 1) + B(u - 1) \\
& 1 = Au + A + Bu - B \\
& 1 = (A + B)u + (A - B) \\
& A + B = 0 \quad A - B = 1 \\
& (1 + B) + B = 0 \quad B = -\frac{1}{2} \quad A = \frac{1}{2} \\
& \frac{1}{(u - 1)(u + 1)} = \frac{1}{2(u - 1)} - \frac{1}{2(u + 1)} \\
& \int \frac{1}{(u - 1)(u + 1)} \, du = \int \frac{1}{2(u - 1)} - \frac{1}{2(u + 1)} \, du \\
& \frac{1}{2} \int \frac{1}{u - 1} - \frac{1}{u + 1} \, du \\
& \frac{1}{2} (\ln |u - 1| - \ln |u + 1|) + C \\
& 2u + C + 2 \int \frac{1}{(u - 1)(u + 1)} \, du = 2u + 2 \left(\frac{1}{2} (\ln |u - 1| - \ln |u + 1|) \right) + C \\
& 2u + \ln |u - 1| - \ln |u + 1| + C
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech