Taylor and Maclaurin Series

Alvin Lin

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Taylor and Maclaurin Series

Here is a function $f(x)$ that can be represented as a power series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n(x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + ... \end{aligned}$$

If we assume that the derivatives of $f(x)$ in every order exist, then we can solve for the coefficients c_n. For example:

$$f(a) = c_0$$

Using the first derivative:

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 \\ f'(a) &= c_1 \end{aligned}$$

Using the second derivative:

$$\begin{aligned} f''(x) &= 2c_2 + (3)(2)c_3(x-a) + (4)(3)(x-a)^2 + (5)(4)(x-a)^3 \\ f''(a) &= 2c_2 \\ c_2 &= \frac{f''(a)}{2} \end{aligned}$$

Using the third derivative:

$$\begin{aligned} f'''(x) &= (3)(2)(1)c_3 + (4)(3)(2)c_4(x-a) + (5)(4)(3)c_5(x-a) \\ f'''(a) &= (3)(2)c_3 \\ c_3 &= \frac{f'''(a)}{(3)(2)} \end{aligned}$$

It follows that a general form of the coefficient c_n is:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Provided a power series representation for the function $f(x)$ about $x = a$ exists, the Taylor Series for $f(x)$ about $x = a$ is:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f'(a)(x-a)^2}{2!} + \frac{f''(a)(x-a)^3}{3!} + ... \end{aligned}$$

If we use $a = 0$, then the Taylor Series is known as the Maclaurin Series for $f(x)$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + ... \end{aligned}$$

Example 1

Find the Taylor Series for $f(x) = \mathrm{e}^x$ about $x = 0$:

$$f(x) = \mathrm{e}^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$\begin{aligned} f^{(n)}(x) &= \mathrm{e}^x \\ f^{(n)}(0) &= \mathrm{e}^x \\ f(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \end{aligned}$$

We can actually derive the value of e by taking $f(0)$:

$$f(0) = \mathrm{e} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + ...$$

Example 2

$$\begin{aligned}
\sin(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... \\
&= 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \\
\cos(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \\
\mathrm{e}^{ix} &= \cos(x) + i\sin(x)
\end{aligned}$$

Example 3

$$\lim_{x\to 0} \frac{1-\cos(x)}{1+x-\mathrm{e}^x}$$

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{1+x-\mathrm{e}^x} &= \lim_{x\to 0} \frac{1-[1-\frac{x^2}{2!}+\frac{x^4}{4!}-...]}{1+x-[1+\frac{x}{1!}+\frac{x^2}{2!}+...]} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-...}{\frac{-x^2}{2!}-\frac{x^3}{3!}} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2}}{\frac{-x^2}{2}}\left[\frac{1-\frac{x^2}{2}-...}{1+\frac{x}{3}}\right] \\
&= -1
\end{aligned}$$

Example 4

$$\lim_{x\to\infty} \frac{\sin(x)-x-x^{\frac{3}{6}}}{x^3}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sin(x)-x-x^{\frac{3}{6}}}{x^3} &= \lim_{x\to 0} \frac{(x-\frac{x^3}{3!}+\frac{x^5}{5!}-...)-x-x^{\frac{3}{6}}}{x^3} \\
&= \lim_{x\to 0} \frac{-1}{3} + \frac{x^2}{5!} - ... \\
&= -\frac{1}{3}
\end{aligned}$$

Example 5

$$y = \mathrm{e}^x \ln(1-x)$$

$$\begin{aligned} y &= \mathrm{e}^x \ln(1-x) \\ &= (1 + \frac{x}{1!} + \frac{x^2}{2!} + ...)(-x - \frac{x^2}{2} + ...) \end{aligned}$$

Example 6

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{\pi}{4})^{2n+1}}{(2n+1)!} \\ &= \frac{\frac{\pi}{4}}{1!} - \frac{(\frac{\pi}{4})^3}{3!} + ... \\ &= \sin(\frac{\pi}{4}) \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

Example 7

$$\begin{gathered} 1 - \ln(2) + \frac{(\ln(2))^2}{2!} - \frac{(\ln(3))^3}{3!} + ... \\ = \mathrm{e}^{-\ln(2)} = \frac{1}{2} \end{gathered}$$

Binomial Series

Binomial Theorem: If n is any positive integer, then:

$$\begin{aligned}
(a+b)^n &= \sum_{i=0}^{\infty} \binom{n}{i} a^{n-i} b^i \\
&= a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + ... + nab^{n-1} + b^n
\end{aligned}$$

$$\begin{aligned}
&\text{where :} \\
&\binom{n}{i} = \frac{n(n-1)(n-2)...(n-i+1)}{i!}; \quad i = 1, 2, 3...n \\
&\binom{n}{0} = 0
\end{aligned}$$

Example 1

$$(1+x)^k \quad |x| < 1$$

$$\begin{aligned}
(1+x)^k &= \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} \binom{k}{n} x \\
&= 1 + \frac{kx}{1!} + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)...(k-(n-1))x^n}{n!}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + ... \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + ...
\end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech