Absolute Convergence and the Ratio and Root Tests

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Absolute Convergence and the Ratio and Root Tests

Definition: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges. If $\sum |a_n|$ diverges but $\sum a_n$ converges, then $\sum a_n$ is conditionally convergent.

The Ratio Test

Let $\sum a_n$ be given series. $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$

1. If $L < 1$, then $\sum a_n$ converges absolutely.
2. If $L > 1$, then $\sum a_n$ diverges.
3. If $L = 1$, then the test fails.

Example 1

$$\sum \frac{n^2}{2^n}$$

$$
\begin{aligned}
a_n &= \frac{n^2}{2^n} \\
a_{n+1} &= \frac{(n+1)^2}{2^{n+1}} \\
\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| &= \lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \\
&= \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} \\
&= \lim_{n\to\infty} (\frac{n^2}{n^2})(\frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}) \\
&= \frac{1}{2}
\end{aligned}
$$

Since $\frac{1}{2} < 1$, $\sum a_n$ is absolutely convergent.

Example 2

$$\sum \frac{(-10)^n}{n!}$$

$$
\begin{aligned}
a_n &= \frac{(-10)^n}{n!} \\
a_{n+1} &= \frac{(-10)^{n+1}}{(n+1)!} \\
\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| &= \lim_{n\to\infty} \frac{\frac{(-10)^{n+1}}{(n+1)!}}{\frac{(-10)^n}{n!}} \\
&= \lim_{n\to\infty} \frac{-10}{n+1} \\
&= 0
\end{aligned}
$$

Since $0 < 1$, $\sum a_n$ is absolutely convergent.

Example 3

$$\sum a_n = 1 - \frac{1\times 3}{3!} + \frac{1\times 3\times 5}{5!} - \frac{1\times 3\times 5\times 7}{7!} + ...$$

$$
\begin{aligned}
a_n &= (-1)^{n-1}\frac{1\times 3\times 5\times ...(2n-1)}{(2n-1)!}\\
a_{n+1} &= (-1)^{n}\frac{1\times 3\times 5\times ...(2n-1)\times(2n+1)}{(2n+1)!}\\
\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| &= \frac{1\times 3\times 5\times ...(2n-1)\times(2n+1)}{(2n+1)!}\times\frac{(2n-1)!}{1\times 3\times 5\times ...(2n-1)}\\
&= \lim_{n\to\infty}\frac{2n+1}{(2n)(2n+1)}\\
&= \lim_{n\to\infty}\frac{1}{2n}\\
&= 0
\end{aligned}
$$

Since $0 < 1$, $\sum a_n$ is absolutely convergent.

Example 4

$$\sum_{n=1}^{\infty}\frac{n^2 2^{n-1}}{(-5)^n}$$

$$
\begin{aligned}
a_n &= \frac{n^2 2^{n-1}}{(-5)^n}\\
a_{n+1} &= \frac{(n+1)^2 2^n}{(-5)^{n+1}}\\
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(n+1)^2 2^n}{(-5)^{n+1}}}{\frac{n^2 2^{n-1}}{(-5)^n}}\right|\\
&= \frac{(n+1)^2 2^n 5^n}{5^{n+1} n^2 2^{n-1}}\\
&= \frac{2}{5}(\frac{n+1}{n})^2\\
&= \frac{2}{5}(1+\frac{1}{n})^2\\
&= \frac{2}{5}
\end{aligned}
$$

Since $\frac{2}{5} < 1$, $\sum a_n$ is absolutely convergent.

The Root Test

Given $\sum a_n$:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L$$

1. If $L < 1$, the series is absolutely convergent.
2. If $L > 1$, the series diverges.
3. If $L = 1$, the test fails.

Example 1

$$\sum_{n=1}^{\infty} (\frac{n^2+1}{2n^2+1})^n$$

$$\begin{aligned} a_n &= (\frac{n^2+1}{2n^2+1})^n \\ \lim_{n\to\infty} \sqrt[n]{(\frac{n^2+1}{2n^2+1})^n} &= \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} \\ &= \lim_{n\to\infty} (\frac{n^2}{n^2})(\frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}}) \\ &= \frac{1}{2} \end{aligned}$$

Since $\frac{1}{2} < 1$, $\sum a_n$ is absolutely convergent.

Example 2

$$\sum_{n=1}^{\infty} (1+\frac{1}{n})^{n^2}$$

$$\begin{aligned} a_n &= (1+\frac{1}{n})^{n^2} \\ \lim_{n\to\infty} \sqrt[n]{(1+\frac{1}{n})^{n^2}} &= \lim_{n\to\infty} (1+\frac{1}{n})^n \\ &= \mathrm{e} \end{aligned}$$

Since $\mathrm{e} > 1$, $\sum a_n$ diverges.

Example 3

$$\sum_{n=1}^{\infty}(\frac{-2n}{n+2})^{7n}$$

$$\begin{aligned} a_n &= (\frac{-2n}{n+2})^{7n} \\ \lim_{n\to\infty}\sqrt[n]{(\frac{-2n}{n+2})^{7n}} &= \lim_{n\to\infty}(\frac{2n}{n+2})^7 \\ &= \lim_{n\to\infty}(\frac{n}{n})^7(\frac{2}{1+\frac{2}{n}})^7 \\ &= 2^7 \end{aligned}$$

Since $2^7 > 1$, $\sum a_n$ diverges.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech