

# Absolute Convergence and the Ratio and Root Tests

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## Absolute Convergence and the Ratio and Root Tests

**Definition:**  $\sum a_n$  is absolutely convergent if  $\sum |a_n|$  converges. If  $\sum |a_n|$  diverges but  $\sum a_n$  converges, then  $\sum a_n$  is conditionally convergent.

### The Ratio Test

Let  $\sum a_n$  be given series.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

1. If  $L < 1$ , then  $\sum a_n$  converges absolutely.
2. If  $L > 1$ , then  $\sum a_n$  diverges.
3. If  $L = 1$ , then the test fails.

### Example 1

$$\sum \frac{n^2}{2^n}$$

$$\begin{aligned}
a_n &= \frac{n^2}{2^n} \\
a_{n+1} &= \frac{(n+1)^2}{2^{n+1}} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \\
&= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{2n^2} \\
&= \lim_{n \rightarrow \infty} \left( \frac{n^2}{n^2} \right) \left( \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{2} \right) \\
&= \frac{1}{2}
\end{aligned}$$

Since  $\frac{1}{2} < 1$ ,  $\sum a_n$  is absolutely convergent.

### Example 2

$$\begin{aligned}
&\sum \frac{(-10)^n}{n!} \\
a_n &= \frac{(-10)^n}{n!} \\
a_{n+1} &= \frac{(-10)^{n+1}}{(n+1)!} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{(-10)^{n+1}}{(n+1)!}}{\frac{(-10)^n}{n!}} \\
&= \lim_{n \rightarrow \infty} \frac{-10}{n+1} \\
&= 0
\end{aligned}$$

Since  $0 < 1$ ,  $\sum a_n$  is absolutely convergent.

### Example 3

$$\sum a_n = 1 - \frac{1 \times 3}{3!} + \frac{1 \times 3 \times 5}{5!} - \frac{1 \times 3 \times 5 \times 7}{7!} + \dots$$

$$\begin{aligned}
a_n &= (-1)^{n-1} \frac{1 \times 3 \times 5 \times \dots (2n-1)}{(2n-1)!} \\
a_{n+1} &= (-1)^n \frac{1 \times 3 \times 5 \times \dots (2n-1) \times (2n+1)}{(2n+1)!} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{1 \times 3 \times 5 \times \dots (2n-1) \times (2n+1)}{(2n+1)!} \times \frac{(2n-1)!}{1 \times 3 \times 5 \times \dots (2n-1)} \\
&= \lim_{n \rightarrow \infty} \frac{2n+1}{(2n)(2n+1)} \\
&= \lim_{n \rightarrow \infty} \frac{1}{2n} \\
&= 0
\end{aligned}$$

Since  $0 < 1$ ,  $\sum a_n$  is absolutely convergent.

#### Example 4

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n} \\
a_n &= \frac{n^2 2^{n-1}}{(-5)^n} \\
a_{n+1} &= \frac{(n+1)^2 2^n}{(-5)^{n+1}} \\
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{(n+1)^2 2^n}{(-5)^{n+1}}}{\frac{n^2 2^{n-1}}{(-5)^n}} \right| \\
&= \frac{(n+1)^2 2^n 5^n}{5^{n+1} n^2 2^{n-1}} \\
&= \frac{2}{5} \left( \frac{n+1}{n} \right)^2 \\
&= \frac{2}{5} \left( 1 + \frac{1}{n} \right)^2 \\
&= \frac{2}{5}
\end{aligned}$$

Since  $\frac{2}{5} < 1$ ,  $\sum a_n$  is absolutely convergent.

## The Root Test

Given  $\sum a_n$ :

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L$$

1. If  $L < 1$ , the series is absolutely convergent.
2. If  $L > 1$ , the series diverges.
3. If  $L = 1$ , the test fails.

### Example 1

$$\sum_{n=1}^{\infty} \left( \frac{n^2 + 1}{2n^2 + 1} \right)^n$$

$$\begin{aligned} a_n &= \left( \frac{n^2 + 1}{2n^2 + 1} \right)^n \\ \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{n^2 + 1}{2n^2 + 1} \right)^n} &= \lim_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 + 1} \\ &= \lim_{n \rightarrow \infty} \left( \frac{n^2}{n^2} \right) \left( \frac{1 + \frac{1}{n^2}}{2 + \frac{1}{n^2}} \right) \\ &= \frac{1}{2} \end{aligned}$$

Since  $\frac{1}{2} < 1$ ,  $\sum a_n$  is absolutely convergent.

### Example 2

$$\sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^{n^2}$$

$$\begin{aligned} a_n &= \left( 1 + \frac{1}{n} \right)^{n^2} \\ \lim_{n \rightarrow \infty} \sqrt[n]{\left( 1 + \frac{1}{n} \right)^{n^2}} &= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n \\ &= e \end{aligned}$$

Since  $e > 1$ ,  $\sum a_n$  diverges.

### Example 3

$$\sum_{n=1}^{\infty} \left(\frac{-2n}{n+2}\right)^{7n}$$

$$\begin{aligned} a_n &= \left(\frac{-2n}{n+2}\right)^{7n} \\ \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{-2n}{n+2}\right)^{7n}} &= \lim_{n \rightarrow \infty} \left(\frac{2n}{n+2}\right)^7 \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{n}\right)^7 \left(\frac{2}{1+\frac{2}{n}}\right)^7 \\ &= 2^7 \end{aligned}$$

Since  $2^7 > 1$ ,  $\sum a_n$  diverges.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)