

# Series

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## Series

Sequence  $\{a_1, \dots, a_n, \dots\} \equiv \{a_n\}$ .

$$\sum a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots = S_n$$

If:  $\lim_{n \rightarrow \infty} S_n = S$

Then:  $\sum a_n$  converges,  $\sum a_n = S$ ,  $\sum a_n < \infty$

## Geometric Series

$$S_n = a + ar + ar^2 + ar^3 + \dots$$

$$rS_n = ar + ar^2 + ar^3 + \dots$$

$$S_n(1 - r) = a - ar^n$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r}$$

$$\lim_{n \rightarrow \infty} S_n = \frac{a}{1 - r}; |r| < 1$$

If  $|r| < 1$ , the series converges, if  $|r| \geq 1$ , the series diverges. If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ . Proof:

$$S_n - S_{n-1} = a_n$$

$$\lim_{n \rightarrow \infty} S_n - S_{n-1} = \lim_{n \rightarrow \infty} a_n$$

$$S - S = \lim_{n \rightarrow \infty} a_n = 0$$

### Problem 23

$$\sum_{k=2}^{\infty} \frac{k^2}{k^2 - 1} = \frac{4}{3} + \frac{9}{8} + \dots$$

$$a_n = \frac{n^2}{n^2 - 1}$$

$$\sum a_n = a_1 + a_1 r + a_1 r^2 + \dots$$

$$\frac{a_3}{a_2} = \frac{a_2}{a_1} = r$$

$$\sum a_n < \infty$$

$$\lim_{n \rightarrow \infty} a_n \rightarrow 1 \therefore \sum a_n \rightarrow \infty \quad (\text{Diverging})$$

### Problem 25

$$\sum \frac{1 + 2^n}{2^n}$$

$$\begin{aligned} \sum \frac{1 + 2^n}{2^n} &= \sum \frac{1}{3^n} + \sum \frac{2^n}{3^n} \\ &= \sum \left(\frac{1}{3}\right)^n + \sum \left(\frac{2}{3}\right)^n \\ &= \frac{a}{1 - r} + \frac{a}{1 - r} \\ &= \frac{\frac{1}{3}}{1 - \frac{1}{3}} + \frac{\frac{2}{3}}{1 - \frac{2}{3}} \\ &= \frac{1}{2} + \frac{2}{1} \\ &= \frac{5}{2} \end{aligned}$$

### Problem 29

$$\sum_{n=1}^{\infty} \ln\left(\frac{n^2 + 1}{2n^2 + 1}\right)$$

$$a_n = \ln\left(\frac{n^2 + 1}{2n^2 + 1}\right)$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \ln\left(\frac{n^2 + 1}{2n^2 + 1}\right) \\
&= \lim_{n \rightarrow \infty} \ln\left(\frac{n^2(1 + \frac{1}{n^2})}{n^2(2 + \frac{1}{n^2})}\right) \\
&= \lim_{n \rightarrow \infty} \ln\left(\frac{1 + \frac{1}{n^2}}{2 + \frac{1}{n^2}}\right) \\
&= \ln\left(\frac{1}{2}\right)
\end{aligned}$$

$$\lim_{n \rightarrow \infty} a_n = \ln\left(\frac{1}{2}\right) \therefore \sum a_n = \infty \quad (\text{Diverges})$$

### Problem 33

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[ \frac{1}{e^n} + \frac{1}{n(n+1)} \right] \\
\sum_{n=1}^{\infty} \left[ \frac{1}{e^n} + \frac{1}{n(n+1)} \right] &= \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\
\sum_{n=1}^{\infty} \frac{1}{e^n} &= \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \dots \\
&= \frac{1}{1-r} \\
&= \frac{\frac{1}{e}}{1 - \frac{1}{e}} \\
&= \frac{1}{e-1} \\
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \\
&= 1 \\
\sum_{n=1}^{\infty} \left[ \frac{1}{e^n} + \frac{1}{n(n+1)} \right] &= \frac{1}{e-1} + 1
\end{aligned}$$

### Problem 35

$$\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$$

$$\begin{aligned} a_n &= \frac{2}{n^2 - 1} \\ &= \frac{2}{(n-1)(n+1)} \\ &= \frac{(n+1) - (n-1)}{(n-1)(n+1)} \\ &= \frac{1}{n-1} - \frac{1}{n+1} \\ \sum_{n=2}^{\infty} a_n &= \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots \\ S_n &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n} \\ \lim_{n \rightarrow \infty} S_n &= \frac{3}{2} \end{aligned}$$

### p-series

$$\sum \frac{1}{n^p}$$

If  $p > 1$ , the series is convergent. If  $p \leq 1$ , the series is divergent.

- $\sum \frac{1}{n}$  diverges.
- $\sum \frac{1}{n^2}$  converges.
- $\sum \frac{1}{n^{\frac{1}{3}}}$  diverges.
- $\sum \frac{1}{n^{-2}}$  diverges.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)