Sequences

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Sequences

A list of numbers written in a definite order. Example:

$$\left\{1, 19, 11, 12, 19, 13, ... \frac{1}{11}, 19^2\right\}$$

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$$

Define a mapping:

$$f : N \to R$$

$$f(n) = a_n$$

$$\left\{a_1, a_2, a_3, a_4, ...a_n\right\}$$

a_n is the generic term where $a_n = \frac{1}{n}$.

Definition

A sequence $\{a_n\}$ has the limit L if we can make the terms as close to L as we want by choosing a sufficiently large n. If a_n has a limit L, we say that $\{a_n\}$ converges to L. We denote this by $a_n \to L$.

If there is no real number L such that $a_n \to L$, then $\{a_n\}$ diverges.

Example:

$$a_n = \frac{1}{n}$$

$$a_n \to 0$$

$$\lim_{n\to\infty} a_n = L$$

- a_n is increasing iff $a_n < a_{n+1} \quad \forall \quad n$.
- a_n is decreasing iff $a_n > a_{n+1} \quad \forall \quad n$.
- a_n is monotonic if it is either increase or decreasing.
- a_n is bounded from above iff $a_n \leq M \quad \forall \quad n$.
- a_n is bounded from below iff $a_n \geq M \quad \forall \quad n$.
- r^n is convergent is $-1 < r \leq 1$ and divergent for all other r's.

Identities

- $\lim ca_n = c \lim a_n$
- $\lim a_n \pm b_n = \lim a_n + \lim b_n$
- $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$
- $\lim a_n b_n = (\lim a_n)(\lim b_n)$
- $\lim ((a_n)^p) = (\lim a_n)^p \quad (p > 0, a_n > 0)$

Theorems

- If $\lim_{x\to\infty} f(x) = L$, then $a_n \to L$ where $a_n = f(n)$.
- If $\lim |a_n| = 0$, then $\lim a_n = 0$.
- Every bounded monotonic sequence is convergent. (Bounded from above and below). $|a_n| < M$.
- The Squeeze Theorem:

$$a_n \leq b_n \leq c_n \quad n \geq n_o$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$
$$\to \lim_{n\to\infty} b_n = L$$

Practice Problem 17

$$a_n = 1 - (0.2)^n$$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (1 - (0.2)^n) \\ &= 1 - \lim_{n\to\infty} (0.2)^n \\ &= 1 - 0 \\ &= 1\end{aligned}$$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 \leq r \leq 1 \\ 1 & r = 1 \end{cases}$$

Any limit of this form converges when $-1 < r \leq 1$ otherwise it diverges.

Practice Problem 18

$$a_n = \frac{n^3}{n^3 + 1}$$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^3}{n^3 + 1} \\ &= \lim_{n\to\infty} \frac{n^3}{n^3 + 1} \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \\ &= \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n^3}} \\ &= \frac{1}{1 + 0} \\ &= 1\end{aligned}$$

Practice Problem 21

$$a_n = \mathrm{e}^{\frac{1}{n}}$$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \mathrm{e}^{\frac{1}{n}} \\ &= \mathrm{e}^{\lim_{n\to\infty} \frac{1}{n}} \\ &= \mathrm{e}^0 \\ &= 1\end{aligned}$$

Practice Problem 25

$$a_n = \frac{(-1)^n n}{n^2+1}$$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} \\
\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \frac{n}{n^2+1} \\
&= \lim_{n\to\infty} \frac{n}{n^2+1}\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} \\
&= \frac{0}{1+0} \\
&= 0
\end{aligned}$$

$$\lim_{n\to\infty} |a_n| = 0 \quad \therefore \quad \lim_{n\to\infty} a_n = 0$$

(Theorem)

Practice Problem 29

$$a_n = \frac{(2n-1)!}{(2n+1)!}$$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} \\
&= \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} \\
&= \lim_{n\to\infty} \frac{1}{(2n+1)(2n)} \\
&= 0
\end{aligned}$$

Practice Problem 36

$$a_n = \ln(n+1) - \ln(n)$$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \ln(n+1) - \ln(n) \\
&= \lim_{n\to\infty} \ln(\frac{n+1}{n}) \\
&= \lim_{n\to\infty} \ln(1+\frac{1}{n}) \\
&= \ln(1) \\
&= 0
\end{aligned}$$

Practice Problem 37

$$a_n = n\sin(\frac{1}{n})$$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{x\to\infty} x\sin(\frac{1}{x}) \\
&= \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} \\
&= \lim_{y\to 0} \frac{\sin(y)}{y} \\
&\text{(L'Hopital's Rule)} \\
&= \lim_{y\to 0} \frac{\cos(y)}{1} \\
&= \lim_{t\to 0} \cos(y) \\
&= 1
\end{aligned}$$

Practice Problem 38

$$a_n = \sqrt[n]{2^{1+3n}}$$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} 2^{\frac{1+3n}{n}} \\
&= \lim_{n\to\infty} 2^{\frac{1}{n}+3} \\
&= 2^{0+3} \\
&= 8
\end{aligned}$$

Practice Problem 39

$$a_n = (1 + \frac{2}{n})^n$$
$$\lim_{n\to\infty} (1 + \frac{2}{n})^n$$
$$y = y(x) = (1 + \frac{2}{x})^x$$
$$\ln(y) = x\ln(1 + \frac{2}{x})$$
$$\ln(y) = \frac{\ln(1 + \frac{2}{x})}{\frac{1}{x}}$$

By L'Hopital's Rule:

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(1 + \frac{2}{x})}{\frac{1}{x}} = 2$$
$$\ln(y) = 2$$
$$y = \mathrm{e}^2$$
$$\lim_{n\to\infty} a_n = \mathrm{e}^2$$

Practice Problem 40

$$a_n = \frac{\sin(2n)}{1 + \sqrt{n}}$$
$$0 \leq a_n \leq \frac{1}{1 + \sqrt{n}}$$

By the Squeeze Theorem, since 0 and $\frac{1}{1+\sqrt{n}}$ converge to 0, a_n must converge to 0.

Practice Problem 66

$$a_n = n + \frac{1}{n}$$

Either $a_n \geq a_{n+1}$ or $a_{n+1} \geq a_n$ for all n.

$$f(x) = x + \frac{1}{x}$$
$$f'(x) = 1 + (-x^2) = 1 - \frac{1}{x^2}$$
$$f'(x) > 0$$

We can conclude that a_n is increasing and has a lower bound of 0. Thus, a_n diverges.

Practice Problem 68

$$a_1 = \sqrt{2}$$
$$a_2 = \sqrt{2 + \sqrt{2}}$$
$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$
$$a_{n+1} = \sqrt{2 + a_n}$$
$$a_{n+2} = \sqrt{2 + a_{n+1}}$$

By the Monotonic Convergence Theorem:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$$

$$\lim_{n\to\infty} a_{n+1} = \sqrt{2 + \lim_{n\to\infty} a_n} = \sqrt{2 + L}$$
$$L^2 = 2 + L$$
$$L^2 - L - 2 = 0$$
$$(L-2)(L+1) = 0$$
$$L = 2 \quad L = -1$$

We can eliminate -1 since we know the bounds of the sequence.

$$\lim_{n\to\infty} a_n = 2$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech