Area of a Surface of Revolution

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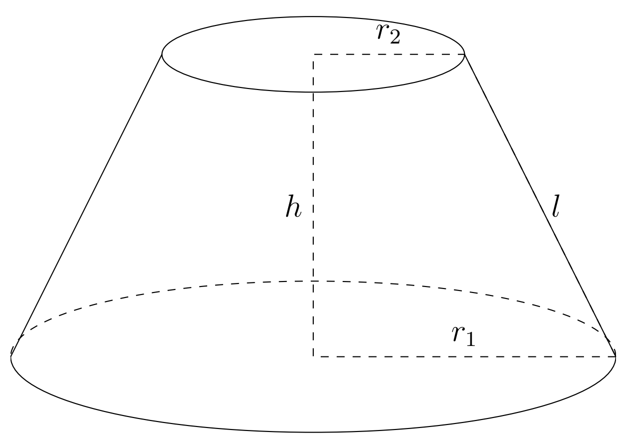
Area of a Surface of Revolution

Finding the surface area of a solid of revolution follows a similar process as finding its volume. Instead of integrating volumes of cross sections, we divide the solid of revolution into frustums and use the arc length formula to integrate the surface areas of the frustums.



$$A = 2\pi r l \quad \text{where} \quad r = \frac{r_1 + r_2}{2}$$

If we have a function $y = f(x)$ being revolved around the x-axis and split into frustums, then:

$$r_1 = f(x_i)$$

$$r_2 = f(x_{i-1})$$

$$l = S \quad (arc\ length) = \sqrt{1 + (f'(x_i))^2}\Delta x$$

We can assume that Δx is infinitely small and $f(x)$ is continuous, thus both $f(x_i)$ and $f(x_{i-1})$ converge to some point $f(x)$. Therefore:

$$r = \frac{f(x_i) + f(x_{i-1})}{2} \approx f(x)$$

$$A = 2\pi f(x)\sqrt{1 + (f'(x))^2}\Delta x$$

If we integrate to find the surface area of the entire solid, it follows that:

$$\begin{aligned} A &= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\ \mathrm{d}x \\ &= \int_a^b 2\pi y\sqrt{1 + (f'(x))^2}\ \mathrm{d}x \end{aligned}$$

This general form can also be modified for rotating a function about the y-axis and also with a function in terms of y.

Rotating about the x and y axes

About x-axis:

$$S = \int (2\pi y)\ \mathrm{d}S$$

About y-axis:

$$S = \int (2\pi x)\ \mathrm{d}S$$

If you have $y = f(x) \quad a \leq x \leq b$:

$$\mathrm{d}S = \int_a^b \sqrt{1 + (\frac{\mathrm{d}y}{\mathrm{d}x})^2}\ \mathrm{d}x$$

If you have $x = g(y) \quad c \leq y \leq d$:

$$\mathrm{d}S = \int_c^d \sqrt{1 + (\frac{\mathrm{d}x}{\mathrm{d}y})^2}\ \mathrm{d}y$$

Example 1

$$y = x^4 \quad 0 \leq x \leq 1$$

$$\begin{aligned} S &= 2\pi \int_0^1 y\sqrt{1 + (y')^2}\ \mathrm{d}x \\ &= 2\pi \int_0^1 x^4\sqrt{1 + 16x^6}\ \mathrm{d}x \end{aligned}$$

Example 2

$$y = x^3 \quad 0 \leq x \leq 2$$

$$\begin{aligned} S &= 2\pi \int_0^2 y\sqrt{1 + (y')^2}\ \mathrm{d}x \\ &= 2\pi \int_0^2 x^3\sqrt{1 + 9x^4}\ \mathrm{d}x \\ &\mathit{Let} : u = 1 + 9x^4 \qquad \mathrm{d}u = 36x^3\ \mathrm{d}x \\ &= 2\pi \int \sqrt{u}\frac{\mathrm{d}u}{36} \\ &= \frac{\pi}{18}\int u^{\frac{1}{2}}\ \mathrm{d}u \\ &= \frac{\pi}{18}\frac{2u^{\frac{3}{2}}}{3} \\ &= \frac{\pi}{27}u^{\frac{3}{2}} \\ &= \frac{\pi}{27}\left[(1 + 9x^4)^{\frac{3}{2}}\right]_0^2 \end{aligned}$$

Example 3

$$y = \tan^{-1}(x) \quad 0 \leq x \leq 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 + x^2}$$

Rotating about the x-axis:

$$
\begin{aligned}
S &= \int_0^1 2\pi y\sqrt{1+(\frac{\mathrm{d}y}{\mathrm{d}x})^2}\ \mathrm{d}x \\
&= \int_0^1 2\pi \tan^{-1}(x)(1+(\frac{1}{1+x^2})^2)^{\frac{1}{2}}\ \mathrm{d}x
\end{aligned}
$$

Rotating about the y-axis:

$$
\begin{aligned}
S &= \int_0^1 2\pi x\sqrt{1+(\ \mathrm{d}y/\ \mathrm{d}x)^2}\ \mathrm{d}x \\
&= \int_0^1 2\pi x(1+\frac{1}{(1+x^2)^2})^{\frac{1}{2}}\ \mathrm{d}x
\end{aligned}
$$

Example 4

Rotated about y-axis:

$$f(x) = y = \frac{x^3}{6} + \frac{1}{2x} \quad \frac{1}{2} \leq x \leq 1$$

$$S = \int 2\pi y\ \mathrm{d}S$$

$$S = \int_{\frac{1}{2}}^1 2\pi y\sqrt{1+(\frac{\mathrm{d}y}{\mathrm{d}x})^2}\ \mathrm{d}x$$

$$y = \frac{x^3}{6} + \frac{1}{2x} \quad y' = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$
\begin{aligned}
1+(\frac{\mathrm{d}y}{\mathrm{d}x})^2 &= 1+(\frac{x^2}{2}-\frac{1}{2x^2})^2 \\
&= 1+\frac{x^4}{4}+\frac{1}{4x^4}-2(\frac{x^2}{2})(\frac{1}{2x^2}) \\
&= 1+\frac{x^4}{4}+\frac{1}{4x^4}-\frac{1}{2} \\
&= (\frac{x^2}{2})^2+(\frac{1}{2x^2})+\frac{1}{2} \\
&= (\frac{x^2}{2})^2+(\frac{1}{2x^2})^2+2(\frac{x^2}{2})(\frac{1}{2x^2}) \\
&= (\frac{x^2}{2}+\frac{1}{2x^2})^2
\end{aligned}
$$

Hence :

$$
\begin{aligned}
S &= \int_{\frac{1}{2}}^{1} 2\pi(\frac{x^3}{6}+\frac{1}{2x})(\frac{x^2}{2}+\frac{1}{2x^2})\ \mathrm{d}x \\
&= 2\pi\int_{\frac{1}{2}}^{1}\left[\frac{x^5}{12}+\frac{x}{12}+\frac{x}{4}+\frac{1}{4x^3}\right]\mathrm{d}x \\
&= 2\pi\int_{\frac{1}{2}}^{1}\left[\frac{x^5}{12}+\frac{x}{3}+\frac{1}{4x^3}\right]\mathrm{d}x \\
&= 2\pi\left[\frac{x^6}{72}+\frac{x^2}{6}+\frac{1}{-8x^2}\right]_{\frac{1}{2}}^{1} \\
&= \frac{263\pi}{256}
\end{aligned}
$$

Example 5

Rotated about y-axis:

$$x=\sqrt{a^2-y^2} \quad 0\le y\le \frac{a}{2}$$

$$S=\int 2\pi x\ \mathrm{d}S$$

$$S=\int_0^{a/2}\sqrt{a^2-y^2}\sqrt{1+(\frac{\mathrm{d}x}{\mathrm{d}y})^2}\ \mathrm{d}x$$

$$x=\sqrt{a^2-y^2} \quad \frac{\mathrm{d}x}{\mathrm{d}y}=\frac{1}{2}(a^2-y^2)^{\frac{-1}{2}}(-2y)$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{-y}{\sqrt{a^2 - y^2}}$$

$$\begin{aligned} 1 + (\frac{\mathrm{d}y}{\mathrm{d}x})^2 &= 1 + \frac{y^2}{a^2 - y^2} \\ &= \frac{a^2 - y^2 + y^2}{a^2 - y^2} \\ &= \frac{a^2}{a^2 - y^2} \end{aligned}$$

Hence :

$$\begin{aligned} S &= \int_0^{a/2} 2\pi\sqrt{a^2 - y^2}(\frac{a^2}{a^2 - y^2})\ \mathrm{d}y \\ &= \int_0^{a/2} 2\pi\sqrt{a^2 - y^2}(\frac{\sqrt{a^2}}{\sqrt{a^2 - y^2}})\ \mathrm{d}y \\ &= \int_0^{a/2} 2\pi a\ \mathrm{d}y \\ &= 2\pi a\Big[y\Big]_0^{a/2} \\ &= 2\pi a(\frac{a}{2} - 0) \\ &= \pi a^2 \end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech