Improper Integrals

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Improper Integrals

$$\int_0^1 x \, \mathrm{d}x$$

This is a proper integral. It represents the area under the curve of the function $y = x$ from 0 to 1.

$$\int_0^\infty x \, \mathrm{d}x$$

This is an example of an improper integral. It has no proper meaning because you cannot take the area under the curve from 0 to ∞. For any integral of this form, the problem is only as hard as is the integration of $f(x)$. We can rewrite it as:

$$\lim_{b\to\infty} \int_a^b f(x) \, \mathrm{d}x$$

Practice Problem 5

$$\int_3^\infty \frac{1}{(x-2)^{3/2}} \, \mathrm{d}x$$

$$\lim_{b\to\infty} \int_3^b \frac{1}{(x-2)^{3/2}} \, \mathrm{d}x$$

$$\lim_{b\to\infty} \int_3^b (x-2)^{-3/2} \, \mathrm{d}x$$

Solve it as an indefinite integral:

$$\int (x-2)^{-3/2} \, \mathrm{d}x$$

$$Let: f'(x) = x^{-3/2}\ \mathrm{d}x \quad g(x) = x - 2$$
$$f(x) = \frac{-2}{\sqrt{x}} \quad g'(x) = \ \mathrm{d}x$$
$$f(g(x)) = \int (x-2)^{-3/2}\ \mathrm{d}x = -\frac{2}{\sqrt{x-1}} + C$$

Evaluate using the original limits:

$$-2 \lim_{b\to\infty} \left[\frac{1}{\sqrt{x-1}}\ \mathrm{d}x\right]_3^b$$
$$-2 \lim_{b\to\infty} \left[\frac{1}{\sqrt{b-2}} - \frac{1}{\sqrt{3-2}}\right]$$
$$-2(0-1)$$
$$= -2$$

Practice Problem 11

$$\int_0^\infty \frac{x^2}{\sqrt{1+x^2}}\ \mathrm{d}x$$
$$\lim_{b\to\infty} \int_0^b \frac{x^2}{\sqrt{1+x^2}}\ \mathrm{d}x$$

Solve it as an indefinite integral:

$$\int \frac{x^2}{\sqrt{1+x^2}}\ \mathrm{d}x$$
$$Let: t = 1 + x^3$$
$$\mathrm{d}t = 3x^2\ \mathrm{d}x$$
$$\int \frac{x^3}{\sqrt{t}} \frac{\mathrm{d}t}{3x^2}$$
$$\frac{1}{3} \int \frac{1}{\sqrt{t}}\ \mathrm{d}t$$
$$\frac{2}{3}\sqrt{t}$$
$$\frac{2}{3}\sqrt{1+x^3}$$

Evaluate using the original limits:

$$\frac{2}{3}\lim_{b\to\infty}\left[\sqrt{1+x^3}\right]_0^b$$

$$\frac{2}{3}\lim_{b\to\infty}\left[\sqrt{1+b^3}-\sqrt{1+0^3}\right]$$

$$=\infty$$

Practice Problem 21

$$\int_0^{\infty}\frac{\ln(x)}{x}\ \mathrm{d}x$$

$$\lim_{b\to\infty}\int_1^b\frac{\ln(x)}{x}\ \mathrm{d}x$$

Solve it as an indefinite integral:

$$\int\frac{\ln(x)}{x}\ \mathrm{d}x$$

$$Let: t=\ln(x)$$

$$\mathrm{d}t=\frac{1}{x}\ \mathrm{d}x$$

$$\int\frac{t}{x}x\ \mathrm{d}t$$

$$\int t\ \mathrm{d}t$$

$$\frac{1}{2}t^2$$

$$\frac{(\ln(x))^2}{2}$$

Evaluate using the original limits:

$$\lim_{b\to\infty}\left[\frac{(\ln(x))^2}{2}\right]_1^b$$

$$\lim_{b\to\infty}\left[\frac{(\ln(b))^2}{2}-\frac{(\ln(1))^2}{2}\right]$$

$$=\infty$$

Review

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\ \mathrm{d}x$$

$$\int_{-\infty}^{0} \frac{x^2}{9+x^6}\ \mathrm{d}x + \int_{0}^{\infty} \frac{x^2}{9+x^6}\ \mathrm{d}x$$

Because $\frac{x^2}{9+x^6}$ is an even function, we can rewrite this as:

$$2\int_{0}^{\infty} \frac{x^2}{9+x^6}\ \mathrm{d}x$$

Solve it as an indefinite integral:

$$2\int \frac{x^2}{9+x^6}\ \mathrm{d}x$$

$$2\int \frac{x^2}{9+(x^3)^2}\ \mathrm{d}x$$

$$Let : 3t = x^3$$

$$3\ \mathrm{d}t = 3x^2\ \mathrm{d}x$$

$$\mathrm{d}t = x^2\ \mathrm{d}x$$

$$2\int \frac{x^2}{9+(3t)^2} x^2\ \mathrm{d}t$$

$$2\int \frac{1}{9+9t^2}\ \mathrm{d}t$$

$$\frac{2}{9}\int \frac{1}{1+t^2}\ \mathrm{d}t$$

$$\frac{2}{9}\tan^{-1}(t) + C$$

$$\frac{2}{9}\tan^{-1}(\frac{x^3}{3}) + C$$

Evaluate using the original limits:

$$\frac{2}{9}\lim_{a\to\infty}\left[\tan^{-1}(\frac{x^3}{3})\right]_0^a$$

$$\frac{2}{9}\lim_{a\to\infty}\left[\tan^{-1}(\frac{a^3}{3})-\frac{0^3}{3}\right]$$
$$\frac{2}{9}\lim_{a\to\infty}\left[\frac{\pi}{2}-0\right]$$
$$=\frac{\pi}{9}$$

Review 2

$$\int_1^\infty \frac{1}{x^{1/2}+x^{3/2}}\ \mathrm{d}x$$
$$\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\ \mathrm{d}x$$

Solve it as an indefinite integral:

$$\int \frac{1}{\sqrt{x}(1+x)}\ \mathrm{d}x$$
$$Let: x=t^2$$
$$\mathrm{d}x = 2t\ \mathrm{d}t$$
$$\int \frac{1}{t(1+t^2)}2t\ \mathrm{d}t$$
$$2\int \frac{1}{1+t^2}\ \mathrm{d}t$$
$$2\tan^{-1}(t)=2\tan^{-1}(\sqrt{x})$$

Evaluate using the original limits:

$$\lim_{a\to\infty}\left[2\tan^{-1}(\sqrt{x})\right]_1^a$$
$$\lim_{a\to\infty}\left[2\tan^{-1}(\sqrt{a})-2\tan^{-1}(1)\right]_1^a$$
$$2(\frac{\pi}{2})-2(\frac{\pi}{4})$$
$$=\frac{\pi}{2}$$

Review 3

$$\int \frac{x+4}{x^2+2x+5}\ \mathrm{d}x$$
$$\int \frac{x+4}{x^2+2x+5}\ \mathrm{d}x$$
$$\int \frac{x+4}{x^2+2x+5}\ \mathrm{d}x$$
$$\frac{1}{2}\int \frac{2x+8}{x^2+2x+5}\ \mathrm{d}x$$
$$\frac{1}{2}\int \frac{2x+2+6}{x^2+2x+5}\ \mathrm{d}x$$
$$\frac{1}{2}\int \frac{2x+2}{x^2+2x+5} + \frac{6}{x^2+2x+5}\ \mathrm{d}x$$
$$\frac{1}{2}\ln|x^2+2x+5| + C + \frac{1}{2}\int \frac{6}{x^2+2x+5}\ \mathrm{d}x$$
$$\frac{1}{2}\int \frac{6}{x^2+2x+5}\ \mathrm{d}x = 3\int \frac{1}{x^2+2x+1+4}\ \mathrm{d}x$$
$$3\int \frac{1}{(x+1)^2+4}\ \mathrm{d}x$$
$$Let : x+1 = 2\tan(\theta)$$
$$\mathrm{d}x = 2\sec^2(\theta)\ \mathrm{d}\theta$$
$$3\int \frac{1}{(2\tan(\theta))^2+4}2\sec^2(\theta)\ \mathrm{d}\theta$$
$$3\int \frac{2\sec^2(\theta)}{4(\tan^2(\theta)+1)}\ \mathrm{d}\theta$$
$$\frac{3}{2}\int \frac{\sec^2(\theta)}{\sec^2(\theta)}\ \mathrm{d}\theta$$
$$\frac{3}{2}\int 1\ \mathrm{d}\theta$$
$$\frac{3}{2}\theta$$
$$x+1 = 2\tan(\theta) \quad \theta = \tan^{-1}(\frac{x+1}{2})$$
$$= \frac{3}{2}\tan^{-1}(\frac{x+1}{2})$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech